Math. Math. in the Appl. Sci., 13, (1990), 305-322.

Dissipative Maxwell's Equations at Low Frequencies

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MOS classi cations:35B30,35B40,35F15,35J05,35Q20,78A25,78A30,78A40
Key words:
Maxwell's equations, conductive media, low frequency, magnetostatics, electrostatics, radiating solutions, transmission problems, exterior problems.

Abstract

A study of the low frequency behavior of solutions to dissipative Maxwell's equations with discontinuous coe cients is given.

1 Introduction

Let $x \in \mathbf{R}^3$, $\omega > 0$,

$$\text{curl } E = i\omega\mu(x)H \tag{1.1}$$

$$\text{curl } H = -i\omega\varepsilon(x)E + \sigma(x)E + j(x,\omega). \tag{1.2}$$

Let us assume that ε , μ and σ are symmetric-matrix-valued functions such that the following de niteness assumptions are satis ed:

$$\sigma(x) \geq \sigma_o > 0 \quad \text{in} \quad D, \quad \sigma(x) = 0 \quad \text{in} \quad \mathbf{R}^3 \setminus \overline{D} \tag{1.3}$$

$$\varepsilon(x) \geq c_1 > 0, \quad \mu(x) \geq c_2 > 0, \quad c_j = \text{const} > 0 \tag{1.4}$$

where D is a bounded domain with the strict cone property. The entries of the matrix σ are in $\mathcal{L}_\infty$ and those of ε and μ are in $\mathcal{C}_2$. Moreover we assume

$$\varepsilon(x) = \varepsilon_o \text{ id} \quad , \quad \mu(x) = \mu_o \text{ id} \quad \text{for} \quad |x| > R_o \tag{1.5}$$

where R_o is an arbitrary large xed number. The function $\sigma(x)$ is discontinuous across $\Gamma := \partial D$. Maxwell's equations (1.1) , (1.2) describe a dissipative system since $\sigma \not\equiv 0$. Let us assume that E and H satisfy the radiation condition at in nity:

$$E(x) = \frac{\exp(i\omega\kappa_o r)}{r} F_E(\theta) + o\left(\frac{1}{r}\right), \quad r \to \infty \quad \theta \cdot F_E(\theta) = 0 \tag{1.6}$$

where $r := |x|, \quad \theta = xr^{-1}$, $\kappa_o = (\varepsilon_o\mu_o)^{1/2}$. The function F_E is called the radiation pattern for the electric eld and a similar radiation condition holds for the magnetic eld. The radiation condition for the magnetic eld is a consequence of (1.6) and the formula

$$H = \left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2} \theta \times E + o(r^{-1}), \quad r \to \infty \tag{1.7}$$

where $\times$ stands for vector product.

Finally, let us assume that

$$\overline{D} \subset \Omega_{R_o} \text{ and } \operatorname{supp} j(x,\omega) \subset \Omega_{R_o} \setminus \overline{D} \tag{1.8}$$

where $\Omega_R := \{x\colon \ |x| < R\}$ and

$$j(x,\omega) = j_o(x) + i\omega j_1(x,\omega) \tag{1.9}$$

De ning $\hat{I}_o := j_o$ and $\hat{J}_\omega := j_1(\cdot,\omega)$ we assume

$$\operatorname{div} \hat{I}_o = 0 \tag{1.10}$$

and suppose that $\hat{J}_\omega$ is continuous from $\omega \in [0,1]$ to $(\mathcal{L}_2(\mathbf{R}^3))^3$. In practice one wants to be able to take $j(x,\omega)$ to be, for example, a loop of current (a magnetic dipole). The boundary conditions across Γ are:

$$N \times E, \ N \times H, \ \mu H \cdot N \text{ and } (\sigma(x) - i\omega\varepsilon(x))\, E \cdot N \text{ are continuous across } \Gamma \tag{1.11}$$

where N is the normal to Γ.

The problem we are interested in is the following one: What is the behavior of E and H as $\omega \to 0$? In the exterior of a perfect conductor and for $\sigma = 0$ this problem has been discussed in [21, 22] (for homogeneous and isotropic media) and for more general media in [10]. In this case (1.1),(1.2) can be treated as a problem for a selfadjoint operator in a suitable Hilbert space. The case $\sigma \not\equiv 0$ leads to a non-selfadjoint operator and was studied in [15]. The discontinuity and semide niteness of σ bring additional di culty. For unbounded domains this case has not been studied in the literature. For bounded domains we refer to [20]. (The results of [20] will be used intensively in the convergence proof.)

The low frequency behavior of solutions to scalar elliptic equations of the second order has been studied in [2, 3, 4, 12, 13, 23, 24]. Electromagnetic wave scattering by small bodies was studied in [14]. Properties of solutions to elliptic and parabolic equations with discontinuous coe cients are discussed in [6, 8, 18].

The di culty in the study of the low frequency behavior of solutions to (1.1), (1.2), (1.6) and (1.11) is that the problem (1.1), (1.2) at $\omega = 0$ is uncoupled: formally one has

$$\text{curl } E_o = 0, \quad \text{curl } H_o = \sigma E_o + j_o \qquad (1.12)$$

where $E_o = E(x, \omega = 0)$, $H_o = H(x, \omega = 0)$, and one needs additional conditions in order to get unique E_o and H_o and to obtain a physically correct formulation of the static problem at $\omega = 0$. This fomulation is:
$E_o = H_o = 0$ in the region D where $\sigma > 0$, (1.12) holds in $\mathbf{R}^3 \setminus \overline{D}$, $N \times E_o = 0$ on Γ, and the body D is electroneutral, that is

$$\int_\Gamma \varepsilon E_o \cdot N ds = 0$$

Finally, E_o and H_o vanish at in nity.

Having introduced some standard notation in section 2 we outline the approach and formulate the result in section 3. In sections 4 and 5 proofs are given. The assumptions about D, $\varepsilon(x)$, $\mu(x)$, $\sigma(x)$ and $j(x, \omega)$ will not be repeated and are assumed to hold unless otherwise stated. The numeration of formulas is autonomous in each of the sections. References to the formulas from other sections contain the number of this section. References to the formulas of the same section do not contain the number of the section.

2 Preliminaries

The following function spaces are common in electromagnetic theory. We use the notations from [7] and let G be a domain in $\mathbf{R}$:

$$\begin{aligned}
&\mathcal{R}(G) := \{E \in \mathcal{L}_2(G)^3 \text{: curl } E \in \mathcal{L}_2(G)^3 \quad \text{in the sense of distributions}\} \\
&\overset{\circ}{\mathcal{R}}(G) := \{E \in \mathcal{R}(G) \text{: } \langle \text{curl } E, H \rangle = \langle E, \text{curl } H \rangle \quad \text{for all } H \in \mathcal{R}(G)\} \\
&\mathcal{R}_o(G) := \{E \in \mathcal{R}(G) \text{: curl } E = 0\}, \quad \overset{\circ}{\mathcal{R}}_o(G) := \overset{\circ}{\mathcal{R}}(G) \cap \mathcal{R}_o(G) \\
&\mathcal{D}(G) := \{E \in \mathcal{L}_2(G)^3 \text{: div } E \in \mathcal{L}_2(G) \quad \text{in the sense of distributions}\} \\
&\overset{\circ}{\mathcal{D}}(G) := \{E \in \mathcal{D}(G) \text{: } \langle \text{div } E, u \rangle = -\langle E, \nabla u \rangle \quad \text{for all } u \in \mathcal{H}_1(G)\} \\
&\mathcal{D}_o(G) := \{E \in \mathcal{D}(G) \text{: div } E = 0\}, \quad \overset{\circ}{\mathcal{D}}_o(G) := \overset{\circ}{\mathcal{D}}(G) \cap \mathcal{D}_o(G) \\
&\mathcal{D}_\varepsilon(G) := \{E \in \mathcal{L}_2(G)^3 \text{: } \varepsilon E \in \mathcal{D}(G)\} = \varepsilon^{-1}\mathcal{D}(G), \quad \mathcal{D}_{\varepsilon,o}(G) := \varepsilon^{-1}\mathcal{D}_o(G) \\
&\overset{\circ}{\mathcal{D}}_\mu(G) := \mu^{-1}\overset{\circ}{\mathcal{D}}(G), \quad \overset{\circ}{\mathcal{D}}_{\mu,o}(G) := \mu^{-1}\overset{\circ}{\mathcal{D}}_o(G).
\end{aligned}$$

We write

$$\begin{aligned}
\|E\|(G) &:= \left(\int_G |E(x)|^2 \, dx\right)^{1/2} \\
\langle E, F \rangle(G) &:= \int_G E(x) \cdot \overline{F(x)} dx
\end{aligned}$$

for the norm and scalar product in

$$\mathcal{L}_2(G)^3 := \{E : G \longrightarrow \mathbf{C}^3 \text{ (measurable)} \text{: } \|E\| \leq \infty\}$$

(The speci cation $'(G)'$ may sometimes be omitted.) For G unbounded we shall need weighted spaces of this kind (cf. [9, 17]) like e.g.

$$\mathcal{L}_2^{\sim} := \{E \in (\mathcal{L}_2^{\mathrm{loc}}(\mathbf{R}^3))^3 \colon \int_{\mathbf{R}^3} (1+|x|)^{-2} \, |E(x)|^2 \, dx < \infty\}$$

$$\mathcal{R}^{\sim} := \{E \in \mathcal{L}_2^{\sim} \colon \text{curl } E \in \mathcal{L}_2(\mathbf{R}^3)^3\}$$

$$\mathcal{H}_1^{\sim} := \{E \in \mathcal{L}_2^{\sim} \colon \partial_n E_m \in \mathcal{L}_2(\mathbf{R}^3)^3\}$$

We shall frequently use certain direct decompositions of $\mathcal{L}_2(G)^3$ and their corresponding projections (c.f. [7, Thm. 8.3]):

Definition 1 By P_ε and $\overset{\circ}{Q}_\varepsilon := id - P_\varepsilon$, id : the identity, we denote the projections of $\mathcal{L}_2(G)^3$ onto $\mathcal{D}_{\varepsilon,o}(G)$ and $\overline{\nabla \overset{\circ}{\mathcal{H}}_1(G)}$ according to the decomposition

$$\mathcal{L}_2(G)^3 \;=\; \mathcal{D}_{\varepsilon,o}(G) \oplus \overline{\nabla \overset{\circ}{\mathcal{H}}_1(G)}. \tag{2.1}$$

Then the adjoints P_ε^* and $\overset{\circ}{Q}{}_\varepsilon^*$ correspond to the decomposition

$$\mathcal{L}_2(G)^3 \;=\; \mathcal{D}_o(G) \oplus \varepsilon \overline{\nabla \overset{\circ}{\mathcal{H}}_1(G)}. \tag{2.2}$$

By $\overset{\circ}{P}_\mu$ and $Q_\mu := id - \overset{\circ}{P}_\mu$ we denote the projections of $\mathcal{L}_2(G)^3$ onto $\overset{\circ}{\mathcal{D}}_{\mu,o}(G)$ and $\overline{\nabla \mathsf{H}_1(G)}$ according to the decomposition

$$\mathcal{L}_2(G)^3 \;=\; \overset{\circ}{\mathcal{D}}_{\mu,o}(G) \oplus \overline{\nabla \mathsf{H}_1(G)}. \tag{2.3}$$

Then the adjoints $\overset{\circ}{P}{}_\mu^*$ and Q_μ^* correspond to the decomposition

$$\mathcal{L}_2(G)^3 \;=\; \overset{\circ}{\mathcal{D}}_o(G) \oplus \mu \overline{\nabla \mathsf{H}_1(G)}. \tag{2.4}$$

Lemma 1 There exists a constant γ depending only on the sup-norms of ε, ε^{-1}, μ, and μ^{-1} such that

i) $P_\varepsilon^* \varepsilon = \varepsilon P_\varepsilon, \overset{\circ}{Q}{}_\varepsilon^* \varepsilon = \varepsilon \overset{\circ}{Q}_\varepsilon, \overset{\circ}{P}{}_\mu^* \mu = \mu \overset{\circ}{P}_\mu$ and $Q_\mu^* \mu = \mu Q_\mu$.

ii) $\gamma^{-1} \|P_\varepsilon E\| \le \|P_{id} E\| \le \gamma \|P_\varepsilon E\|$

iii) $\gamma^{-1} \|Q_\mu E\| \le \|Q_{id} E\| \le \gamma \|Q_\mu E\|$

iv) If G has the segment property then in Definition 1 the closures $\overline{\nabla \overset{\circ}{\mathcal{H}}_1(G)}$ and $\overline{\nabla \mathcal{H}_1(G)}$ may be replaced by $\nabla \overset{\circ}{\mathcal{H}}_1(G)$ resp. $\nabla \mathcal{H}_1(G)$ if G is bounded and by $\nabla \overset{\circ}{\mathcal{H}}{}_1^{\sim}(G)$ and $\nabla \mathcal{H}_1^{\sim}(G)$ if G is unbounded.

Proof: i) follows by straightforward computation. For ii) consider

$$\begin{aligned} E &= \varepsilon^{-1} F + G \\ E &= F_o + G_o \end{aligned}$$

where $F_o, F \in \mathcal{D}_o(G)$ and $G_o, G \in \overline{\nabla \overset{\circ}{\mathcal{H}}_1(G)}$. These imply

$$\langle F_o \,,\, F \rangle = \langle \varepsilon^{-1} F \,,\, F \rangle$$

whence

$$\|\varepsilon^{-1}F\| = \|P_\varepsilon E\| \le \gamma\|F_o\| = \gamma\|P_{id}E\|$$

The other half of ii) and iii) are proved similarly. iv) is a result of R.Picard [9], see also [7]. q.e.d.

In the sequel B always denotes a xedboundedopen set containing Ω_{R_o}. The following sets will be convenient occasionally (cf. $(??)$):

$$\begin{array}{ccc} \Omega & := & \Omega_R \\ A & := A_R := & \Omega \setminus \bar{D} \\ \dot{B} & := & B \setminus \bar{D} \\ \dot{\Omega} & := & \Omega \setminus \bar{D} \end{array} \tag{2.5}$$

We also de ne

$$\varepsilon' := \varepsilon + \frac{i\sigma}{\omega} \tag{2.6}$$

3 Formulation of the Result

Firstly let us give a rigorous de nition of the expected static limiting elds (E_o, H_o) . We have

Theorem 1 Consider $\hat{I}_o$, $\hat{J}_o \in \mathcal{L}_2(\mathbf{R}^3)^3$ such that supp $\hat{I}_o$, $\hat{J}_o \subset \Omega_{R_o} \setminus \overline{D}$ and div $\hat{I}_o = 0$. There exists uniquely $(E_o, H_o) \in \mathcal{R}^\sim \times \mathcal{R}^\sim$ such that

$$\text{curl } E_o = 0 \tag{3.1}$$

$$E_o|_D = 0 \tag{3.2}$$

$$\langle \varepsilon E_o - \hat{J}_o \,,\, \nabla w \rangle = 0 \qquad \text{for any } w \in \overset{\bullet}{\mathcal{H}}{}_1^{\sim} := \{w \in \mathcal{H}_1^\sim :\ \nabla w = 0 \text{ in } D\} \tag{3.3}$$

$$\text{curl } H_o = \hat{I}_o \tag{3.4}$$

$$\text{div } (\mu H_o) = 0 \tag{3.5}$$

We shall refer to (E_o, H_o) as 'the solution to the static problem'.
Remark: Generalized solutions in the sense of (3.1) (3.5) are introduced and discussed in [20] in the case of a bounded domain Ω (instead of $\mathbf{R}^3$). It is shown there that (3.1) (3.5) are indeed correct generalizations of (1.11),(1.12).

Theorem 1 may be proved by observing that it is just a special case of [9]. However results for more general static problems in the case of boundeddomains are obtained in [20]. These can be carried over to the present 'unbounded' situation if (following [9]) we make appropiate replacements like $\Omega \to \mathbf{R}^3$, $A \to \mathbf{R}^3 \setminus \overline{D}$, $\overset{\circ}{\mathcal{H}}_1(\Omega) \to \mathcal{H}_1^\sim$, $\overset{\bullet}{\mathcal{H}}_1(A) \to \overset{\bullet}{\mathcal{H}}{}_1^{\sim}(A)$, $\mathcal{L}_2(\Omega)^3 \to \mathcal{L}_2(\mathbf{R}^3)^3$ etc. Note also that the usual Poincaré inequality has to be replaced by 'Poincaré's inequality IV' [7, p.62].
For further reference let us note

Lemma 2 The solutions (E_o, H_o) to the static problem belong to $\mathcal{L}_2(\mathbf{R}^3)^3$. In fact they satisfy (with a constant C independent of $\hat{I}_o, \hat{J}_o$ and $R \geq R_o$) :

i) $\|E_o\|(\mathbf{R}^3) \leq C\|\hat{J}_o\|$

ii) $\|H_o\|(\mathbf{R}^3) \leq C\|\hat{I}_o\|$

iii) $\sup\{|E_o(x)| : |x| = R\} \leq CR^{-2}\|\hat{J}_o\|$

iv) $\sup\{|H_o(x)| : |x| = R\} \leq CR^{-2}\|\hat{I}_o\|$

Proof: For $|x| > R_o$, E_o is a harmonic vector field and hence satisfies $\Delta E_o = 0$ and has an expansion in spherical harmonics:

$$E_o(x) = \sum_{n=0}^{\infty} |x|^{-n-1}\, S_n(\frac{x}{|x|})$$

(The S_n are vector-valued function with spherical harmonics of degree n as components. Note that $E_o \in \mathcal{L}_2^{\sim}$ implies that only decaying components are present.) From

$$\text{div } E_o = \sum_{n=0}^{\infty} \text{div } [|x|^{-n-1}\, S_n(\frac{x}{|x|})] = 0$$

we get

$$\text{div } (|x|^{-1}\, S_o(\frac{x}{|x|})) = 0$$

since the sum is uniformly convergent and its terms are homogeneous of distinct degrees. But $S_o(...) = a$ where a is a constant vector and

$$\text{div } (|x|^{-1}\, a) = -\,|x|^{-3}\,\langle\, x\,,\, a\,\rangle = 0$$

implies $a = 0$. Hence

$$E_o(x) = |x|^{-2} \sum_{n=1}^{\infty} |x|^{-n+1}\, S_n(\frac{x}{|x|})$$

Noting that the sup-norm of the sum may be estimated by $\|E_o\|(\mathcal{L}_2^{\sim})$ we get iii) and hence i). The same argument proves iv) and ii). q.e.d.

Theorem 2 Let $\operatorname{supp} j(\cdot\,,\omega) \subset \Omega_{R_o} \setminus \overline{D}$. There exists a unique solution $(E_\omega, H_\omega) \in \mathcal{R}^{loc}(\mathbf{R}^3) \times \mathcal{R}^{loc}(\mathbf{R}^3)$ of (I.1), (I.2), (I.6). This solution belongs to $\mathcal{R}^{\sim} \times \mathcal{R}^{\sim}$.

We shall refer to this solution as the solution of the time harmonic problem

Remarks: Since $\varepsilon(x),\ \mu(x)$ are constant multiples of the unit matrix for $|x| > R_o$, E_ω and H_ω are smooth there. Hence it makes sense to formulate the radiation condition as in (1.6). The generalized analogues of the transmission conditions (1.11) are contained in the distributional formulation of the problem.

The proof of this theorem will be given in section 4 by limiting absorption. For exterior boundary value problems with $\sigma = 0$ such a proof is carried out in [7]. In our case some adjustments must be made due to the presence of the conductivity and the fact that we do not suppose $j(\cdot\,,\omega) \in \mathcal{D}(\mathbf{R}^3)$. See also [15].

Theorem 3
$$(E_\omega, H_\omega) \to (E_o, H_o) \text{ in } (\mathcal{L}_2^{\text{loc}}(\mathbf{R}^3))^3$$
More precisely for any bounded domain $B \subset \mathbf{R}^3$:

i) If $j_1(\cdot, \omega)$ is bounded in $\mathcal{L}_2(\mathbf{R}^3)^3$ then

$$\begin{aligned} \|E_\omega - E_o\|(D) &\leq C\omega^{1/2} \\ \|H_\omega - H_o\|(B) &\leq C\omega^{1/10} \\ \|E_\omega\|(\dot{B}) &\leq C \end{aligned}$$

ii) If in addition $\hat{J}_\omega \to \hat{J}_o$ in $\mathcal{L}_2(\mathbf{R}^3)^3$ then $E_\omega \to E_o$ in $\mathcal{L}_2(\mathbf{R}^3)^3$. Namely

$$\|E_\omega - E_o\|(\dot{B}) \leq C_1\|\hat{J}_\omega - \hat{J}_o\| + C_2\omega^{1/2}$$

Theorem 3 will be proved in section 5.

4 Proof of existence and uniqueness for xed frequency

In this section $\omega > 0$ is xed. Moreover if in the notations of various spaces, norms, and scalar products no domain is indicated, it is assumed that they refer to whole $\mathbf{R}^3$.

1. Proof of uniqueness: Let (E, H) denote a solution of the homogeneous time harmonic equation. Multiply (1.1) by $\overline{H}$ (the bar stands for complex conjugate), the complex conjugate of (1.2) by E, subtract the second of the resulting equations from the rst, integrate over the ball Ω_r, apply Gauss' theorem to the left side and obtain

$$\int_{S^r} \theta \cdot E \times \overline{H} ds = i\omega \int_{\Omega_r} \left(\mu|H|^2 - \varepsilon|E|^2\right) dx - \int_{\Omega_r} \sigma|E|^2 dx \tag{4.1}$$

where $S_r = \{x\colon \ |x| = r\}$. Use (1.7) and (1.8) to get

$$\theta \cdot E \times \overline{H} = |E|^2 - |\theta \cdot E|^2 + o\left(r^{-2}\right). \tag{4.2}$$

Substitute (4.2) into (4.1) and take real part of (4.1) to get

$$\text{Re}\left\{\int_{S_r} \left(|E|^2 - |\theta \cdot E|^2\right) ds + o(1)\right\} = -\int_D \sigma|E|^2 dx, \quad r \to \infty. \tag{4.3}$$

Since $|E|^2 - |\theta \cdot E|^2 \geq 0$, and $\sigma \geq 0$, it follows from (4.3) that the integral on the right side of (4.3) vanishes. Since $\sigma > 0$ in D one concludes that $E = 0$ in D, and hence (E, H) can be considered as a solution to

$$\begin{aligned} \text{curl } E - \mathrm{i}\omega\mu H = 0 \\ \text{curl } H + \mathrm{i}\omega\varepsilon E = 0 \end{aligned}$$

in $\mathbf{R}^3$ which vanishes in D. Therefore $(E, H) = 0$ by the principle of unique continuation ([7]).

2. Proof of existence: The existence proof is carried out in three steps.

Step 1: In the rst step it is shown that for any $\eta > 0$ and $j \in \mathcal{L}_2^3$ there exists a unique solution in $\mathcal{R} \times \mathcal{R}$ to the problem of nding E_η, H_η such that

$$\begin{aligned} \text{curl } E_\eta - \mathrm{i}\,\omega_\eta \mu H_\eta &= 0 \qquad (4.4) \\ \text{curl } H_\eta + \mathrm{i}\,\omega_\eta \varepsilon' E_\eta &= j \qquad (4.5) \\ \text{with} \qquad \omega_\eta &:= \omega + \mathrm{i}\,\eta. \end{aligned}$$

As in [20] we eliminate E_η from the system and see that (4.4),(4.5) is equivalent with the problem of nding H_η such that for all $\Phi \in \mathcal{R}$

$$\langle\, {\varepsilon'}^{-1} \text{curl } H_\eta \,,\, \text{curl } \Phi \,\rangle - \omega_\eta^2 \langle\, \mu H_\eta \,,\, \Phi \,\rangle \;=\; \langle\, {\varepsilon'}^{-1}\, j_\eta \,,\, \text{curl } \Phi \,\rangle. \qquad (4.6)$$

Then E_η is given by

$$E_\eta := {\varepsilon'}^{-1} \frac{\mathrm{i}}{\omega_\eta} (\text{curl } H_\eta - j) \qquad (4.7)$$

We shall write B_η for the left hand side of (4.6) and consider it as a continuous sesquilinear form on $\mathcal{R}$. B_η is strongly coercive on $\mathcal{R}$, namely for $\Phi \in \mathcal{R}$

$$\begin{aligned} -\mathrm{Im}\, B_\eta(\Phi, \Phi) &\geq \gamma_1 \eta \|\Phi\|^2 \\ \mathrm{Re}\, B_\eta(\Phi, \Phi) &\geq \gamma_2 \|\text{curl } \Phi\|^2 - \gamma_3 \|\Phi\|^2 \qquad (4.8) \end{aligned}$$

hold where $\gamma_1, \gamma_2, \gamma_3$ do not depend on η. By the Lax Milgram theorem the rst step is done.

Step 2: Consider a family $(j_\eta)_{\eta \in (\,0\,,\,1\,]}$ of right hand sides in $\mathcal{L}_2^3$ with $\mathrm{supp}\, j_\eta \subset \Omega_{R_o}$ and converging to j in $\mathcal{L}_2^3$ as η tends to 0. With some $a \in (\,\frac{1}{2}\,,\,1\,]$ we introduce

$$\mathcal{L}_a := \{ U \in (\mathcal{L}_2^{loc})^3 \colon\; \|U\|_a := \|(1+r)^{-a}\, U\| < \infty \}$$

Under the assumption

$$\sup_{0<\eta\leq 1} (\|E_\eta\|_a + \|H_\eta\|_a) \leq c < \infty. \qquad (4.9)$$

it is proved in this step that E_η and H_η converge in $\mathcal{L}_a$ to limits E and H respectively and that this pair (E, H) is the solution of the time harmonic problem with right hand side j.

If (4.9) holds then there exists a sequence $\eta_n \to 0$ such that

$$H_n := H_{\eta_n} \rightharpoonup H, \quad E_n := E_{\eta_n} \rightharpoonup E, \quad n \to \infty \qquad (4.10)$$

where $\rightharpoonup$ denotes weak convergence in $\mathcal{L}_a$. It follows from (4.4) that

$$\text{div } (\mu H_n) = 0 \qquad (4.11)$$

and that curl H_n is bounded in $\mathcal{L}_a$. Hence for any bounded domain $D_2 \subset \mathbf{R}^3$

$$\| H_n \| (D_2) + \| \text{div } \mu H_n \| (D_2) + \| \text{curl } H_n \| (D_2) \leq c \qquad (4.12)$$

with some c depending on D_2 but not on n. This and the inequality

$$\begin{aligned} \| H_n \|_{\mathcal{H}_1(D_1)} \;\leq\;& c(D_1, D_2) \{ \| H_n \| (D_2) + \| \text{div } (\mu H_n) \| (D_2) \\ &+ \| \text{curl } H_n \| (D_2) \} \qquad (4.13) \end{aligned}$$

imply that by Rellich's imbedding theorem one has

$$H_n \to H \quad \text{in} \quad (\mathcal{L}_2^{loc})^3. \tag{4.14}$$

Inequality (??) holds if $D_1 \subset D_2$ is a strictly inner subdomain of D_2 and μ is Lipschitz continuous and positive de nite as can be seen from [16]. However, convergence in $(\mathcal{L}_2^{loc})^3$ could also be derived directly from inequality (4.13), since by [11] the inclusion of $\mathcal{R}(D_2) \cap \mathcal{D}_\mu(D_2)$ into $\mathcal{L}_2(D_1)^3$ is compact, provided that μ is uniformly positive de nite, bounded, and measurable.

Denote by $\chi \in \overset{\circ}{\mathcal{C}}_\infty$ a nonnegative cut-o function which is identically 1 in some bounded domain D_1. With some positive constant γ and for any $n, m \in \mathbf{N}$

$$\begin{aligned}
&\gamma\|\text{curl } H_n - \text{curl } H_m\|^2(D_1) \\
&\leq \text{Re}\langle \varepsilon'^{-1}\text{curl } H_n - \varepsilon'^{-1}\text{curl } H_m \,,\, \chi\text{curl } (H_n - H_m) \rangle \\
&= \text{Re}\Big[\langle \varepsilon'^{-1}\text{curl } H_n \,,\, \text{curl } (\chi(H_n - H_m)) \rangle \\
&\quad -\langle \varepsilon'^{-1}\text{curl } H_m \,,\, \text{curl } (\chi(H_n - H_m)) \rangle\Big] \\
&\quad -\langle \varepsilon'^{-1}\text{curl } (H_n - H_m) \,,\, \nabla\chi \times (H_n - H_m) \rangle
\end{aligned} \tag{4.15}$$

holds. Then because of (4.6) (with $\Phi := \chi(H_n - H_m)$) also curl H_n and by (4.4), (4.5) E_n and curl E_n converge in $(\mathcal{L}_2^{loc})^3$. The limits E, H are in $\mathcal{R}^{loc}$ and satisfy (1.1),(1.2) .

Since for $|x| > R_o$ the components of E_n and H_n are $\mathcal{L}^2$ solutions of the damped Helmholtz equation

$$\Delta u + k_n^2 u = 0 \;, \qquad k_n = \omega_{\eta_n}\kappa_o \tag{4.16}$$

one may use the Green formula to represent E_n and H_n in $\mathbf{R}^3 \setminus \Omega_r$, $r > R_o$ and obtains, for example:

$$E_n(x) = \int_{S_r} \left[E_n(s)\frac{\partial g_n(x,s)}{\partial N} - g_n(x,s)\frac{\partial E_n}{\partial N} \right] ds, \quad r > R_o \tag{4.17}$$

where

$$g_n(x,y) := \frac{\exp(ik_n|x-y|)}{4\pi|x-y|}, \tag{4.18}$$

and N is the normal to S_r pointing into $\mathbf{R}^3 \setminus \Omega_r$. Because of the well known elliptic estimates for the solutions of (4.16) E_n and $\partial E_n/\partial N$ converge to E and $\partial E/\partial N$ uniformly on S_r.

Hence from (4.17) it follows that E satis es the radiation condition and (E, H) is a solution of the time harmonic problem with right hand side j.

Moreover (4.17) and similar representations for E_n, H and H_n show that

$$\| E_n - E \|_a + \| H_n - H \|_a \to 0 \quad \text{as} \quad n \to \infty. \tag{4.19}$$

The uniqueness of the constructed solution (E, H) guarantees that any subsequence (E_{η_n}, H_{η_n}) converges to the same $(E; H)$ in $\mathcal{L}_a$. Thus $E_\eta \to E,\ H_\eta \to H$ in $\mathcal{L}_a$ as $\eta \to 0$.

Step 3: In order to complete the proof one has to show (4.9) . Assume that (4.9) is not true. Then there is a sequence $\eta_n \to 0$ such that

$$\| E_{\eta_n} \|_a + \| H_{\eta_n} \|_a := \alpha_n > n.$$

Divide equation (4.5) by α_n and let

$$(E_n, H_n) := (\frac{1}{\alpha_n} E_{\eta_n}, \frac{1}{\alpha_n} H_{\eta_n}), \quad \|E_n\|_a + \|H_n\|_a = 1. \tag{4.20}$$

Apply to (E_n, H_n) the above argument and obtain that

$$\| E_n - E \|_a + \| H_n - H \|_a \to 0 \tag{4.21}$$

where (E, H) is the unique solution of the time harmonic equation with right hand side $\lim_{n\to\infty} \frac{1}{\alpha_n} j_{\eta_n} = 0$. Hence E and H vanish. This, (4.20) and (4.21) lead to a contradiction which proves (4.9) .

5 Proof of convergence

Let us start with a rough explanation of the basic ideas. In [20] corresponding boundary value problems in a bounded domain are studied using coercivity estimates for the magnetic component uniform with respect to ω. Using re ned Poincaré inequalities these estimates can be carried over to domains Ω_R where R increases moderately while ω decreases. The re nement comes from balancing optimally the following two e ects: Poincaré inequalities improve when lower parts of the spectrum are excluded. On the other hand, lower eigenfunctions satisfy sharper regularity estimates (see (5.8), (5.9) below) and the Poincaré constant in small subdomains improves.

Depending on ω we consider

$$\Omega := \Omega_R := \{x: \quad |x| < R := R(\omega) := q\omega^{-1}\}$$

with xed q (to be described later) and recall the de nitions for ε' and for the sets $D, B, A, ...$ (see (1.8), (2.5) and (2.6)). For nonnegative functions $f_\omega,\ g_\omega$ and numbers $a_\omega,\ b_\omega$ which depend on the frequency ω we shall use the shorthand notations

$$f_\omega(x) \quad \prec \quad g_\omega(x) \quad \text{for } x \in S \qquad (\text{resp.} \quad a_\omega \quad \prec \quad b_\omega\ )$$

which are to be read as follows:
There exist $c, \omega_o \in \mathbf{R}^+$ such that

$$f_\omega(x) \le c\, g_\omega(x) \quad \text{for all } x \in S \qquad (\text{resp.} \quad a_\omega \le c\, b_\omega\ )$$

for $\omega \in (0, \omega_o]$.

Our analysis will be based on two re ned Poincaré type estimates:

Lemma 3 The following estimates hold uniformly with respect to $\Phi \in \overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega) \cap \mathcal{R}(\Omega)$ and with respect to $\Phi \in \mathcal{D}_{\varepsilon,o}(\Omega) \cap \overset{\circ}{\mathcal{R}}(\Omega)$:

i) $\omega^{2/5}\|\Phi\|(B) + \omega\|\Phi\|(A) \quad \prec \quad \|\text{curl } \Phi\|(\Omega)$

ii) $\|\Phi\|(\Omega) \quad \prec \quad \omega^{-2/5}\|\text{curl } \Phi\|(B) + \omega^{-1}\|\text{curl } \Phi\|(A)$

Proof: Transform Ω_R into Ω_1 by $x \longmapsto R^{-1}x$. Let $B_1 := R^{-1}B$, $A_1 := R^{-1}A$ and observe

$$\text{meas}(B_1) \quad \prec \quad \omega^3 \tag{5.1}$$

The function spaces $\overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega_R) \cap \mathcal{R}(\Omega_R)$ and $\overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega_R)$ are transformed into

$$Y := \overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega_1) \cap \mathcal{R}(\Omega_1)$$
$$X := \overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega_1)$$

which we equip with the scalar products

$$\langle\, \Phi\, ,\, \Psi\, \rangle_Y \quad := \quad \langle\, \text{curl } \Phi\, ,\, \text{curl } \Psi\, \rangle(\Omega_1)$$

$$\langle\, \Phi\, ,\, \Psi\, \rangle_X \quad := \quad \langle\, \mu\Phi\, ,\, \Psi\, \rangle(\Omega_1)$$

and the corresponding norms. $\langle\ ,\ \rangle_Y$ is a scalar product because (cf. [9])

$$\lambda_1 := \inf\{\|\text{curl } \Phi\|^2(\Omega_1)\colon\ \Phi \in Y\ ,\ \langle\, \Phi\, ,\, \Phi\, \rangle_X = 1\} > 0. \tag{5.2}$$

Y, X are Hilbert spaces and Y is compactly imbedded in X . Therefore by standard results of functional analysis we get a sequence

$$0 < \lambda_1 \leq \ldots \leq \lambda_k \to \infty$$

and a complete orthonormal system

$$\{\Psi_k\colon\ k \in \mathbf{N}\}$$

in X such that

$$\langle\, \text{curl } \Psi_k\, ,\, \text{curl } \Phi\, \rangle = \lambda_k \langle\, \mu\Psi_k\, ,\, \Phi\, \rangle \qquad \text{for } \Phi \in Y \tag{5.3}$$

In particular with $\mathcal{A}\psi := \text{curl curl } \psi - \text{grad div}\,(\mu\psi)$ (note: $\Psi_k \in \overset{\circ}{\mathcal{D}}_{\mu,o}$):

$$\langle\, \Psi_k\, ,\, \Psi_l\, \rangle_Y = \lambda_k \delta_{kl} \tag{5.4}$$

$$\mathcal{A}\Psi_k = \lambda_k \mu \Psi_k \tag{5.5}$$

For $\Lambda \in [1, \infty)$ we define

$$U := U_\Lambda := \text{span}\{\Psi_k\colon\ \lambda_k \leq \Lambda\}$$

which is a finite-dimensional subspace of both X and Y . Note that

$$X \ominus U \quad = \overline{\text{span}\{\Psi_k\colon\ \lambda_k > \Lambda\}} \quad (\text{closure in } X)$$

$$V := \quad Y \ominus U \quad = \overline{\text{span}\{\Psi_k\colon\ \lambda_k > \Lambda\}} \quad (\text{closure in } Y)$$

$$\|v\|_Y^2 \geq \Lambda \|v\|_X^2 \qquad \text{for } v \in V \tag{5.6}$$

On the other hand for $u \in U$ we get (with a bound C for the eigenvalues of $\mu(x)$)

$$\|\mathcal{A}u\|(\Omega_1) \leq C^{1/2}\Lambda\|u\|_X \tag{5.7}$$

and therefore by regularity theory [19]

$$\|u\|_{\mathcal{H}^2(\Omega_1)} \quad \prec \quad \Lambda\|u\|_X \tag{5.8}$$

Using an interpolated version of Sobolev's imbedding theorem [1, Lemma 13.1] we obtain

$$\sup\{|u(x)| : x \in B_1\} \quad \prec \quad \|u\|_X + \|u\|_X^{1/4}\|u\|_{\mathcal{H}^2(\Omega_1)}^{3/4} \quad \prec \quad \Lambda^{3/4}\|u\|_X \tag{5.9}$$

and using (1)

$$\|u\|(B_1) \quad \prec \quad \omega^{3/2}\Lambda^{3/4}\|u\|_X \quad \prec \quad \omega^{3/2}\Lambda^{3/4}\lambda_1^{-1/2}\|u\|_Y \tag{5.10}$$

Take $\Phi \in Y$, write

$$\Phi = u + v \qquad u \in U \text{ , } v \in V$$

and combine (5.6) and (5.10) to obtain

$$\begin{aligned} \|\Phi\|(B_1) \quad & \leq \quad \|u\|(B_1) + \|v\|(B_1) \\ & \prec \quad \omega^{3/2}\Lambda^{3/4}\|u\|_Y + \Lambda^{-1/2}\|v\|_Y \end{aligned}$$

Observing

$$\|u\|_Y^2 + \|v\|_Y^2 = \|\Phi\|_Y^2 \tag{5.11}$$

we get for $\Lambda := \omega^{-6/5}$

$$\|\Phi\|(B_1) \quad \prec \quad \omega^{3/5}\|\Phi\|_Y$$

Combine this with (5.2) and transform back to Ω_R . In this transformation each of the integrals in i) gets a common factor by change of variables whereas the integral on the right hand side gets an extra factor ω^{-1} by the chain rule . This proves i).

For ii) we de ne

$$\tilde{\mathcal{A}} := \text{curl } \mu^{-1}\text{curl } - \text{grad div}$$

and note that (5.5) implies

$$\tilde{\mathcal{A}}\,\text{curl } \Psi_k = \lambda_k \text{curl } \Psi_k \tag{5.12}$$

We recall the argument that led to (10), replace $u \in U$ by curl u and $\mathcal{A}$ by $\tilde{\mathcal{A}}$ and use (5.12) instead of (5.5) . We obtain analogously

$$\|\text{curl } u\|(B_1) \quad \prec \quad \omega^{3/2}\Lambda^{3/4}\|\text{curl } u\|(\Omega_1) \tag{5.13}$$

With some small (but xed) $c_o \in \mathbf{R}^+$ let us put now

$$\Lambda := c_o\omega^{-6/5}$$

With this Λ (5.13) implies

$$\|\text{curl } u\|(B_1) \quad \prec \quad \omega^{3/5}\|\text{curl } u\|(A_1) \tag{5.14}$$

and

$$
\begin{aligned}
\|\Phi\|_X^2 &= \|u\|_X^2 + \|v\|_X^2 \\
&\leq \lambda_1^{-1}(\|\text{curl } u\|^2(B_1) + \|\text{curl } u\|^2(A_1)) \\
&\quad +\Lambda^{-1}(\|\text{curl } v\|^2(B_1) + \|\text{curl } v\|^2(A_1)) \\
&\prec \|\text{curl } u\|^2(A_1) + \|\text{curl } v\|^2(A_1) + \Lambda^{-1}\|\text{curl } v\|^2(B_1) \\
&=: S
\end{aligned}
\tag{5.15}
$$

(5.16)

Observing that by (5.4)

$$\langle \text{ curl } u \,, \text{ curl } v \,\rangle(A_1) = -\langle \text{ curl } u \,, \text{ curl } v \,\rangle(B_1) \quad \text{for } u \in U, \; v \in V$$

we nd

$$
\begin{aligned}
S &\prec \|\text{curl } \Phi\|^2(A_1) + 2\,|\langle \text{ curl } u \,, \text{ curl } v\,\rangle(B_1)|\, + \Lambda^{-1}\|\text{curl } v\|^2(B_1) \\
&\prec \|\text{curl } \Phi\|^2(A_1) + \Lambda\|\text{curl } u\|^2(B_1) + \Lambda^{-1}\|\text{curl } v\|^2(B_1) \\
&\prec \|\text{curl } \Phi\|^2(A_1) + c_o\|\text{curl } u\|^2(A_1) + \Lambda^{-1}\|\text{curl } v\|^2(B_1)
\end{aligned}
$$

Recalling the de nition of S we see that the second term on the right may be omitted if c_o is chosen small enough. Thus summing up we get

$$\|\Phi\|_X^2 \quad \prec \quad \|\text{curl } \Phi\|^2(A_1) + \omega^{6/5}\|\text{curl } \Phi\|^2(B_1)$$

Transforming back to Ω_R as above we obtain ii). The result for $\mathcal{D}_{\varepsilon,o}(\Omega) \cap \overset{\circ}{\mathcal{R}}(\Omega)$ is proved in exactly the same way. q.e.d.

Using Lemma 3 and recalling $\Omega = \{x\colon \; |\,x\,| < q\omega^{-1}\}$ we can show as in [20] :

Lemma 4 Let $X := \overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega) \cap \mathcal{R}(\Omega)$. For small q the sesquilinear forms

$$
\begin{array}{llll}
b_\omega & : & X \times X & \longrightarrow & \mathbf{C} \\
& & (H, \Phi) & \longmapsto & \langle\, \varepsilon'^{-1}\text{curl } H \,, \text{ curl } \Phi\,\rangle - \omega^2\langle\, \mu H \,, \Phi\,\rangle
\end{array}
$$

are strongly coercive in the following sense:

$$\omega\|\text{curl } \Phi\|^2(D) + \|\text{curl } \Phi\|^2(\dot{\Omega}) \quad \prec \quad |\, b_\omega(\Phi, \Phi)\,|$$

uniformly w.r.t. $\Phi \in X$.

>From now on q is a xed positive number such that Lemma 4 is applicable. We recall a representation formula for radiating solutions of homogeneous isotropic Maxwell equations:

Lemma 5 Let $(E,H)\in(\mathcal{L}^{2,\text{loc}})^3\times(\mathcal{L}^{2,\text{loc}})^3$ satisfy the radiation condition (I.6,7) and

$$\text{curl } E - \mathrm{i}\,\omega\,\mu_o H = F \tag{5.17}$$

$$\text{curl } H + \mathrm{i}\,\omega\,\varepsilon_o E = G \tag{5.18}$$

where $F,G\in\mathcal{L}_2(\mathbf{R}^3)^3$ and supp F , supp $G\subset B$. Then for $x\notin B$:

$$E(x)=\int_B(\mathrm{i}\,\omega\,\mu_o G(y)g(x,y)+F(y)\times\nabla_y g(x,y)+\frac{1}{\mathrm{i}\,\omega\,\varepsilon_o}D^2g(x,y)G(y))dy$$

where

$$g(x,y):=\frac{1}{4\pi\,|x-y|}\exp(\mathrm{i}\,\omega\,\kappa_o\,|x-y|)$$

and

$$D^2g(x,y):=\left[\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}g(x,y)\right]$$

denotes the matrix of second derivatives.

We are now ready for the

Proof (of Theorem 3): Recalling $R=q\omega^{-1}$ we see that we have for $y\in B$ and $x\in Z_R:=\Omega_R\setminus\bar{\Omega}_{R-1}$

$$|\,\partial_x^\alpha\partial_y^\beta g(x,y)\,|\quad\prec\quad\omega^{1+|\alpha|+|\beta|} \tag{5.19}$$

for $x\in Z_R:=\Omega_R\setminus\bar{\Omega}_{R-1}$ and $y\in B$. Using (5.19) we nd (uniformly with respect to radiating solutions of (5.17), (5.18))

$$\sup\{|\,\partial^\alpha E(x)\,|:\,x\in\bar{Z}_R\}\quad\prec\quad\omega^{2+|\alpha|}(\|F\|+\|G\|) \tag{5.20}$$

$$\|E\|(Z_R)+\|\text{curl }(\chi_R E)\|(Z_R)\quad\prec\quad\omega(\|F\|+\|G\|) \tag{5.21}$$

$$\|\text{curl } E\|(Z_R)\quad\prec\quad\omega^2(\|F\|+\|G\|) \tag{5.22}$$

We just used cut-o functions χ_R de ned by

$$\chi_R(x):=\tilde{\chi}(|\,x\,|-R)$$
$$\tilde{\chi}\text{ (xed) }\in C^\infty(\mathbf{R}^1)$$
$$0\le\tilde{\chi}\le1\quad;\quad\tilde{\chi}\,|_{(-\infty\,,\,-1]}=0\quad;\quad\tilde{\chi}\,|_{[0\,,\,\infty)}=1$$

Let us introduce

$$F:=F_\omega:=\mathrm{i}\,\omega\,(\mu-\mu_o)H_\omega$$
$$G:=G_\omega:=\sigma E_\omega-\mathrm{i}\,\omega\,(\varepsilon-\varepsilon_o)E_\omega+\hat{I}_o+\mathrm{i}\,\omega\,\hat{J}_\omega$$

It is then clear that (E_ω,H_ω) and (F_ω,G_ω) satisfy the assumptions of Lemma 5. Therefore using (5.20) (5.22):

$$\sup\{|\,E_\omega(x)\,|:\,x\in\bar{Z}_R\}\quad\prec\quad\omega^2(\|F_\omega\|+\|G_\omega\|) \tag{5.23}$$

$$\|E_\omega\|(Z_R)+\|\text{curl }(\chi_R E_\omega)\|\quad\prec\quad\omega(\|F_\omega\|+\|G_\omega\|) \tag{5.24}$$

$$\|\text{curl } E_\omega\|(Z_R)\quad\prec\quad\omega^2(\|F_\omega\|+\|G_\omega\|) \tag{5.25}$$

After these preparations let us introduce in Ω (cf. Lemmata 2, 1)

$$\tilde{E}:=\tilde{E}_\omega:=(1-\chi_R)E_\omega$$
$$\tilde{H}:=\tilde{H}_\omega:=\overset{\circ}{P}_\mu(H_\omega-H_o)$$

$(\tilde{E},\tilde{H}) \in \overset{\circ}{\mathcal{R}}(\Omega) \times (\overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega) \cap \mathcal{R}(\Omega))$ satis es the following Maxwell system in Ω :

$$\text{curl } \tilde{E} - \mathrm{i}\,\omega\,\mu\tilde{H} = \overset{\circ}{P}{}^*_\mu K \tag{5.26}$$

$$\text{curl } \tilde{H} + \mathrm{i}\,\omega\,\varepsilon'\tilde{E} = J \tag{5.27}$$

where

$$K := K_\omega := -\text{curl } (\chi_R E_\omega) + \mathrm{i}\,\omega\,\mu H_o \tag{5.28}$$

$$J := J_\omega := \mathrm{i}\,\omega\,\hat{J}_\omega - \mathrm{i}\,\omega\,\varepsilon_o\chi_R E_\omega \tag{5.29}$$

>From (5.26),(5.27) we obtain by partial integration

$$\begin{aligned} b_\omega(\tilde{H},\tilde{H}) &= \langle \varepsilon'^{-1}J\,,\,\text{curl } \tilde{H}\,\rangle - \mathrm{i}\,\omega\,\langle\,\overset{\circ}{P}{}^*_\mu K\,,\,\tilde{H}\,\rangle \\ &= \langle \varepsilon'^{-1}J\,,\,\text{curl } \tilde{H}\,\rangle - \mathrm{i}\,\omega\,\langle\,K\,,\,\tilde{H}\,\rangle \end{aligned}$$

Noting that supp $J \subset \dot{\Omega}$ we get from Lemma 4 and Lemma 3ii :

$$\begin{aligned} &\|\text{curl } \tilde{H}\|^2(\dot{\Omega}) + \omega\|\text{curl } \tilde{H}\|^2(D) \\ &\quad\prec \quad \|J\|(\Omega)\cdot\|\text{curl } \tilde{H}\|(\dot{\Omega}) + \omega\|K\|(\Omega)\cdot\|\tilde{H}\|(\Omega) \\ &\quad\prec \quad \|J\|(\Omega)\cdot\|\text{curl } \tilde{H}\|(\dot{\Omega}) \\ &\qquad\quad + \|K\|(\Omega)\left[\omega^{3/5}\|\text{curl } \tilde{H}\|(D) + \|\text{curl } \tilde{H}\|(\dot{\Omega})\right] \end{aligned}$$

This implies

$$\omega^{1/2}\|\text{curl } \tilde{H}\|(D) + \|\text{curl } \tilde{H}\|(\dot{\Omega}) \quad \prec \quad \|J\|(\Omega) + \|K\|(\Omega) =: \alpha \tag{5.30}$$

and (by Lemma 3)

$$\|\tilde{H}\|(B) \quad \prec \quad \omega^{-9/10}\alpha \tag{5.31}$$

$$\|\tilde{H}\|(\Omega) \quad \prec \quad \omega^{-1}\alpha \tag{5.32}$$

We use (5.27) to estimate $E_\omega\,|_B = \tilde{E}\,|_B$:

$$\|E_\omega\|(D) \quad \prec \quad \omega^{-1/2}\alpha \tag{5.33}$$

$$\|E_\omega\|(\dot{B}) \quad \prec \quad \omega^{-1}\alpha \tag{5.34}$$

It remains to estimate $\nabla u := Q_\mu(H_\omega - H_o)$. We have

$$\mu^{-1}\text{curl } \tilde{E} - \mathrm{i}\,\omega\,(H_\omega - H_o) = \mathrm{i}\,\omega\,H_o - \mu^{-1}\text{curl } (\chi_R E)$$

and therefore

$$Q_\mu(H_\omega - H_o) = -Q_\mu H_o + \frac{1}{\mathrm{i}\,\omega}Q_\mu\mu^{-1}\text{curl } (\chi E) \tag{5.35}$$

∇u is characterized by

$$\begin{aligned} \langle\,\mu\nabla u\,,\,\nabla\phi\,\rangle = \langle\,\mu V\,,\,\nabla\phi\,\rangle \qquad &\text{for all } \phi \in \mathcal{H}_1(\Omega) \\ \text{where} \quad V := -H_o + \tfrac{1}{\mathrm{i}\,\omega}\mu^{-1}\text{curl } (\chi_R E)& \end{aligned} \tag{5.36}$$

Using the seminorm

$$|\,u\,|_1 := \|\nabla u\|$$

and the corresponding quotient space

$$\mathcal{H}_1(\Omega)/\{\text{constants}\} \cong \{u \in \mathcal{H}_1(\Omega) : \int_\Omega u(x)dx = 0\} =: \dot{\mathcal{H}}_1(\Omega)$$

we see that (5.36) has a continuous solution operator. Since div$(\mu V) = 0$ we find from (5.36) :

$$\langle \mu V , \nabla\phi \rangle = \int_S \mu_o \theta V \phi(s) ds \quad , S := \partial\Omega_R$$

Using Lemma 2 and (21) we find :

$$\begin{aligned} |\langle \mu V , \nabla\phi \rangle| &\leq \|\phi\|_{\mathcal{L}_2(S)} (\text{meas}(S))^{1/2} \sup\{|\mu_o V(s)| : s \in S\} \\ &\prec \|\phi\|_{\mathcal{L}_2(S)} \omega(\|F\| + \|G\| + \|\hat{I}_o\|) \end{aligned} \tag{5.37}$$

Transforming back to the standard domain Ω_1 and using the trace theorem and a standard version of Poincaré's inequality [7, p.27 'Poincaré's estimate II'] we get

$$\begin{aligned} \|\phi\|_{\mathcal{L}_2(S)} &\prec \omega^{-1/2} \|\nabla\phi\|(\Omega) \\ &\text{uniformly w.r.t. } \phi \in \dot{\mathcal{H}}_1(\Omega) \end{aligned} \tag{5.38}$$

Combining (5.37) and (5.38) we get

$$\|Q_\mu(H_\omega - H_o)\| \prec \omega^{1/2}(\|F\| + \|G\| + \|\hat{I}_o\|) \tag{5.39}$$

Let us now recall the definitions of J, K, F and G :

$$\begin{aligned} \alpha := \|J\| + \|K\| &\prec \omega\|\hat{J}_\omega\| + \omega\|\chi_R E_\omega\| \\ &\quad + \omega\|\mu H_o\| + \|\text{curl } \chi_R E_\omega\| \\ &\prec \omega(\|\hat{J}_\omega\| + \|\hat{I}_o\|) + \omega(\|F\| + \|G\|) \end{aligned} \tag{5.40}$$

$$\begin{aligned} \beta := (\|F\| + \|G\|) &\prec \omega\|H_\omega\|(B) + \|E_\omega\|(D) + \\ &\quad \omega\|E_\omega\|(\dot{B}) + \|\hat{I}_o\| + \omega\|\hat{J}_\omega\| \\ &\prec \omega\|H_\omega - H_o\|(B) + \|E_\omega\|(D) \\ &\quad + \omega\|E_\omega\|(\dot{B}) + \|\hat{I}_o\| + \omega\|\hat{J}_\omega\| \end{aligned} \tag{5.41}$$

Combining (5.40),(5.41) with the estimates (5.31), (5.32),(5.33),(5.34) and (5.39) we find

$$\begin{aligned} \omega^{-1/10}\|H_\omega - H_o\|(B) &\prec \omega^{-1}\alpha + \omega^{2/5}(\beta + \|\hat{I}_o\|) \\ &\prec \|\hat{J}_\omega\| + \|\hat{I}_o\| + \beta \\ &\prec \|\hat{J}_\omega\| + \|\hat{I}_o\| + \omega\|H_\omega - H_o\|(B) \\ &\quad + \|E_\omega\|(D) + \omega\|E_\omega\|(B) =: \gamma \end{aligned} \tag{5.42}$$

$$\|H_\omega - H_o\|(\Omega) \prec \omega^{-1}\alpha + \omega^{1/2}(\beta + \|\hat{I}_o\| \prec \gamma \tag{5.43}$$

$$\|H_\omega\|(\Omega) \prec \gamma \quad \text{(cf. Lemma 2)} \tag{5.44}$$

$$\omega^{-1/2}\|E_\omega\|(D) \prec \omega^{-1}\alpha \prec \gamma \tag{5.45}$$

$$\|E_\omega\|(\dot{B}) \prec \omega^{-1}\alpha \prec \gamma \tag{5.46}$$

Adding (41)-(45) and recalling the de nition of γ we nd

$$\begin{aligned} \|H_\omega - H_o\|(B) &\leq C\omega^{1/10}(\|\hat{I}_o\| + \|\hat{J}_\omega\|) && (5.47)\\ \|E_\omega\|(D) &\leq C\omega^{1/2}(\|\hat{I}_o\| + \|\hat{J}_\omega\|) && (5.48)\\ \|E_\omega\|(\dot{B}) &\leq C(\|\hat{I}_o\| + \|\hat{J}_\omega\|) && (5.49)\\ \gamma &\prec \|\hat{J}_\omega\| + \|\hat{I}_o\| =: \hat{\gamma} && (5.50)\\ \beta = \|F\| + \|G\| &\prec \|\hat{I}_o\| + \omega^{1/2}\|\hat{J}_\omega\| \prec \hat{\gamma} && (5.51) \end{aligned}$$

Up to now we have shown that H_ω and $E_\omega\,|_D$ approach the asserted limits with the rate $\omega^{1/10}$ resp. $\omega^{1/2}$ provided $\hat{J}_\omega$ remains bounded in $\mathcal{L}_2(\Omega)^3$ as $\omega \to 0$. We have also shown that $E_\omega\,|_{\dot{B}}$ remains bounded in this limit process. Thus Theorem 3 i) is proved.

Using compactness arguments in the usual way we could show that $E_\omega\,|_B$ approaches the asserted limit in $\mathcal{L}_2(B)^3$ provided $\hat{J}_\omega \to \hat{J}_o$ in $\mathcal{L}_2$ since we have existence and uniqueness for the limit problem. However we can show directly that $E_\omega\,|_B \to E_o\,|_B$ in $\mathcal{L}^2$ - even with a rate if $\hat{J}_\omega \to \hat{J}_o$ with a rate.

So let us rede ne $\tilde{E}, J, K$ by

$$\tilde{E} := (1-\chi_R)(E_\omega - E_o) \tag{5.52}$$

$$K := (1-\chi_R)\mathrm{i}\,\omega\,\mu H_\omega - \nabla\chi_R \times (E_\omega - E_o) \tag{5.53}$$

$$J := \mathrm{i}\,\omega\,(\hat{J}_\omega - \hat{J}_o) - \mathrm{i}\,\omega\,(\varepsilon E_o - \hat{J}_o) - \mathrm{i}\,\omega\,\varepsilon_o\chi_R(E_\omega - E_o) \tag{5.54}$$

to obtain

$$\text{curl } \tilde{E} = K \tag{5.55}$$

$$\text{curl } (H_\omega - H_o) + \mathrm{i}\,\omega\,\varepsilon\tilde{E} - \sigma\tilde{E} = J \tag{5.56}$$

$$(\tilde{E}, H_\omega - H_o) \in \overset{\circ}{\mathcal{R}}(\Omega) \times \mathcal{R}(\Omega) \tag{5.57}$$

Replacing $\tilde{E}$ by $P_\varepsilon\tilde{E} \in \overset{\circ}{\mathcal{R}}(\Omega) \cap \mathcal{D}_{\varepsilon,o}(\Omega)$ (cf. (5.57)) does not change (54) . Therefore (by (5.44), (5.21),(5.49),(5.51))

$$\|\text{curl } (P_\varepsilon\tilde{E})\| \prec \omega\|H_\omega\| + \|E_\omega\|(Z_R) + \|E_o\|(Z_R) \prec \omega\hat{\gamma} \tag{5.58}$$

Using Lemma 3i we nd

$$\|P_\varepsilon\tilde{E}\|(B) \prec \omega^{3/5}\hat{\gamma} \tag{5.59}$$

It remains to estimate $\overset{\circ}{Q}_\varepsilon\tilde{E}$. By formula (4.20) of [20] $\overset{\circ}{Q}_\varepsilon\tilde{E}$ can be estimated by

$$\|\overset{\circ}{Q}_\varepsilon\tilde{E}\| \prec \omega^{-1}\|QJ\|(\dot{\Omega}) + \|\tilde{E}\|(D) + \|P_\varepsilon\tilde{E}\|(D) \tag{5.60}$$

where Q denotes the orthogonal projection of $\mathcal{L}_2(\Omega)^3$ onto

$$\nabla\,\overset{\bullet}{\mathcal{H}}_1(\Omega)\,,\ \overset{\bullet}{\mathcal{H}}_1(\Omega) := \{u \in \overset{\circ}{\mathcal{H}}_1(\Omega)\colon\ \nabla u = 0 \text{ in } D\}.$$

By (3.3) and (5.58)

$$\begin{aligned} \omega^{-1}\|QJ\|(\dot{\Omega}) &\prec \|\hat{J}_\omega - \hat{J}_o\| + \|E_\omega\|(Z_R) + \|E_o\|(Z_R)\\ &\prec \|\hat{J}_\omega - \hat{J}_o\| + \omega\hat{\gamma}. && (5.61) \end{aligned}$$

Insertion of (5.61),(5.45) and (5.59) into (5.60) then yields

$$\|\overset{\circ}{Q}_\varepsilon\tilde{E}\| \prec \|\hat{J}_\omega - \hat{J}_o\| + \omega^{1/2}\hat{\gamma}$$

q.e.d.

Note added in proof

In proof reading the authors realized that the proof of Lemma 3 holds true in the case of constant coe cients ϵ, μ only. This is because there is no uniform control of the constant appearing in the regularity estimate. To obtain the result for non-constant coe cients we may without loss of generality assume that $\mu_0 = 1$ and $\mu = id$ in $\mathcal{R}^3 \setminus B$. We decompose $\Phi \in \mathcal{R}(\Omega) \cap \overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega)$ as

$$\Phi = \Psi + \nabla\psi \quad , \quad \Psi \in \mathcal{R}(\Omega) \cap \overset{\circ}{\mathcal{D}}_o(\Omega) \quad , \quad \psi \in H^1(\Omega) \; .$$

Then the estimates (i) and (ii) of Lemma 3 hold for Ψ, and since curl $\Psi =$ curl Φ we may replace Ψ by curl Φ on the right hand sides. By $\Phi \in \overset{\circ}{\mathcal{D}}_{\mu,o}(\Omega)$ we have for $\nabla\psi$

$$\langle\, \mu\nabla\psi \, , \, \nabla\psi \,\rangle(\Omega) = -\langle\, \mu\Psi \, , \, \nabla\psi \,\rangle(\Omega) = \langle\, (id - \mu)\Psi \, , \, \nabla\psi \,\rangle(\Omega) \; .$$

But $id - \mu$ is supported in B, whence

$$\|\nabla\psi\|(\Omega) \leq c\|\Psi\|(B) \prec \omega^{-2/5}\|\text{curl } \Phi\|(\Omega) \; .$$

This and a similar argument in the case of the electric boundary condition ($\Phi \in \overset{\circ}{\mathcal{R}}(\Omega) \cap \mathcal{D}_{\varepsilon,o}(\Omega)$ proves the Lemma.

Acknowledgements

This work was supported by the Deutsche Forschungsgemeinschaft via the Sonderforschungsbereich 256 at the University of Bonn.
A.G.R. thanks ONR and NSF for support and R. Picard and E. Somersalo for discussions.

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