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LETTER TO THE EDITOR

Stability of the numerical method for solving the 3D inverse scattering problem with fixed-energy data

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Abstract. Let $|A_\delta(\theta'\theta) - A(\theta',\theta)| < \delta$, where $\delta > 0$ is a small given number, $A(\theta',\theta)$ is the unknown scattering amplitude at a fixed $k > 0$ corresponding to a potential $q(x) \in L^2(B_a)$, $q(x) = \overline{q(x)}$, $q(x) = 0$ in $\Omega_a := \mathbb{R}^3 \setminus B_a$, $B_a := \{x\colon |x| \leqslant a\}$, $\theta', \theta \in S^2$, the unit sphere. Functions $q_\delta(x)$ are constructed, given $\delta > 0$ and the function $A_\delta(\theta',\theta)$, such that $\|q_\delta - q\|_{L^2(B_a)} \to 0$ as $\delta \to 0$. The function $A_\delta(\theta',\theta)$ is not a scattering amplitude but is a measurement of the scattering amplitude, i.e. the noisy scattering data.

Let

$$l_q u - k^2 u = 0 \quad \text{in } \mathbb{R}^3 \qquad l_q u := -\nabla^2 + q(x) \tag{1}$$

$$u = u_0 + v \qquad u_0 := \exp(\mathrm{i}k\theta \cdot x) \tag{2}$$

$$v = A(\theta',\theta)\frac{\exp(\mathrm{i}kr)}{r} + \mathrm{o}\left(\frac{1}{r}\right) \qquad \text{as } r := |x| \to \infty \quad \theta' = x/r. \tag{3}$$

We assume throughout that

$$k > 0 \qquad \text{is fixed} \tag{4}$$

and

$$q = \bar{q} \qquad q \in L^2(B_a) \qquad q = 0 \quad \text{in} \quad \Omega_a := \mathbb{R}^3 \setminus B_a \tag{5}$$

where $B_a := \{x\colon x \in \mathbb{R}^3, |x| \leqslant a\}$, $a > 0$ is a positive number. The function $A(\theta',\theta)$ is the scattering amplitude corresponding to $q(x)$.

The inverse problem (IP) is: given $A(\theta',\theta)$ for all $\theta',\theta \in S^2$ find $q(x)$ which satisfies (5).

It is well known that the direct problem (1), (2) and (3), which consists in finding the scattering solution $u(x,\theta,k)$ given $q(x)$ and $\theta \in S^2$, has a solution and the solution is unique (e.g. see [1]). It is proved in [2–5] that the IP has at most solution. In [3] a numerical method is given for computing $q(x)$ given the exact data $A(\theta',\theta)$ (see also [5] and [6]).

The objective of this paper is to demonstrate that the method given in [3] can be used for stable recovery of $q(x)$ given noisy data $A_\delta(\theta',\theta)$. This means that if for any $\delta > 0$ a function $A_\delta(\theta',\theta)$ is known such that

$$\max_{\theta,\theta' \in S^2} |A(\theta',\theta) - A_\delta(\theta',\theta)| < \delta \tag{6}$$

then one can compute $q_\delta(x)$ such that

$$\|q(x) - q_\delta(x)\|_{L^2(B_a)} \leqslant \eta(\delta) \to 0 \qquad \text{as } \delta \to 0. \tag{7}$$

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In this letter we formulate the method first given in [3] and give a construction of $q_\delta(x)$ such that (7) holds. In [7] a general statement is proved, which shows that if an algorithm exists to find the unique solution of the problem given the exact data then a stable approximation of the solution can be constructed given noisy data.

First we seek to recover $q(x)$ given the exact data. Let q satisfy (5) and $Q(\theta', \theta)$ be the corresponding scattering amplitude at a fixed $k > 0$. Then the scattering solution u, which solves (1)–(3), can be written as

$$u(x, \alpha) = \exp(ik\alpha \cdot x) + \sum_{j=0}^{\infty} A_j(\alpha) Y_j(\theta') h_j(kr)$$
$$\text{for } r := |x| > a \quad \theta' := x/r. \tag{8}$$

Here $\alpha \in S^2$, $Y_j(\theta')$ are the orthonormalised in $L^2(S^2)$ spherical harmonics, $h_j(kr)$ are the spherical Hankel functions normalised so that $h_j(r) \sim r^{-1}\exp(r)$ as $r \to \infty$,

$$A_j(\alpha) := \int_{S^2} A(\beta, \alpha)\overline{Y_j(\beta)}\, d\beta. \tag{9}$$

Formula (8) was used in [3] and [8]. It follows from the fact that the right-hand side of (8) solves equation (1) with $q = 0$ in the region Ω_a and that the difference between the scattering solution $u(x, \alpha)$ and the right-hand side of (8) is $o(r^{-1})$ as $r \to \infty$. Since this difference solves the Helmholtz equation in Ω_a and is $o(r^{-1})$, it has to vanish in Ω_a [9, p 25]. Thus (8) follows.

It is proved in [3] that the function $\tilde{q}(p) := \int q(x) \exp(ip \cdot x)\, dx$, $\int := \int_{\mathbb{R}^3}$, can be computed by the formula

$$\tilde{q}(p) = -4\pi \lim_{\substack{|\theta| \to \infty \\ \theta - \theta' = p, \theta, \theta' \in M}} \left\{ \lim_{\varepsilon \to 0} \int_{S^2} A(\theta', \alpha) v_\varepsilon(\alpha, \theta)\, d\alpha \right\}. \tag{10}$$

Here $M := \{\theta : \theta \in C^3, \theta \cdot \theta = 1\}$, $\theta \cdot \theta' := \theta_1\theta_1' + \theta_2\theta_2 + \theta_3\theta_3'$, $p \in R^3$ is an arbitrary vector, and the sequence $v_\varepsilon := v_\varepsilon(\alpha) := v_\varepsilon(\alpha, \theta)$ is constructed as follows (we suppress the variable θ since θ will be fixed in the construction of $v_\varepsilon(\alpha, \theta)$). To construct v_ε one solves the variational problem

$$\mathscr{F}_\varepsilon(v) := \left\| \exp(-ik\theta \cdot x) \int_{S^2} v(\alpha) u(x, \alpha)\, d\alpha - 1 \right\|_{a_1, \varepsilon} = \min \qquad a_1 > a \tag{11}$$

where $\theta \in M$ is fixed, $|\theta| >> 1$, $u(x, \alpha)$ is given by (8) in Ω_a, $v \in L^2(S^2)$, and

$$\|f\|^2_{a_1, \varepsilon} := \int_{a_1 \leqslant |x| \leqslant \varepsilon^{-1}} |f|^2 (1 + |x|)^{-\gamma}\, dx \qquad 1 < \gamma < 2 \quad \varepsilon^{-1} > a_1. \tag{12}$$

We will take $\gamma = 3/2$. Since $\mathscr{F}_\varepsilon(v) \geqslant 0$ the infimum d_ε of $\mathscr{F}_\varepsilon(v)$ is non-negative, $d_\varepsilon := \inf \mathscr{F}_\varepsilon(v) \geqslant 0$. Let v_n be a minimising sequence: $\mathscr{F}_\varepsilon(v_n) \to d_\varepsilon$ as $n \to \infty$. Denote by $v_\varepsilon = v_\varepsilon(\alpha, \theta)$ any member of the minimising squence such that

$$\mathscr{F}_\varepsilon(v_\varepsilon) \leqslant d_\varepsilon + 1. \tag{13}$$

It is proved in [3] that, as $\varepsilon \to 0$, $\lim_{\varepsilon \to 0} d_\varepsilon \leqslant c$, where $c > 0$ is a constant, $c \leqslant c_1|\theta|^{-1}$ if $\theta \in M$ and $|\theta| >> 1$, where $c_1 > 0$ does not depend on θ, c_1 depends on a and on $\|q\|_{L^2(B_a)}$. Furthermore, any sequence $v_\varepsilon = v_\varepsilon(\alpha, \theta)$ which satisfies (13) can be used in (10).

Next we seek a stable solution of the inverse problem. Assume now that a function $A_\delta(\theta', \theta)$ is known for all $\theta', \theta \in S^2$ such that (6) holds and $\delta > 0$ is a given number which can be chosen arbitrary small. The function $A_\delta(\theta', \theta)$ is not necessarily a scattering amplitude. It is a measurement of the exact scattering data $A(\theta', \theta)$.

The problem is to construct $q_\delta(x)$ such that (7) holds given $A_\delta(\theta', \theta)$ for all $\theta', \theta \in S^2$ and assuming that $\delta > 0$ is known.

Define

$$u_\delta(x, \alpha) := \exp(ik\alpha \cdot x) + \sum_{j=0}^{N} A_{\delta j}(\alpha) Y_j(\theta') h_j(kr) \tag{14}$$

where $N = N(\delta)$ will be chosen later and $A_{\delta j}(\alpha) := (A_\delta(\beta, \alpha), Y_j(\beta))_{L^2(S^2)}$. Let

$$\mathscr{F}_{\delta\varepsilon}(v) := \left\| \exp(-ik\theta \cdot x) \int_{S^2} u_\delta(x, \alpha) v(\alpha)\, \mathrm{d}\alpha - 1 \right\|_{a_{1,\varepsilon}} = \min \qquad a_1 > a. \tag{15}$$

Note that

$$\mathscr{F}_{\delta\varepsilon}(v) = \left\| \exp(-ik\theta \cdot x) \int_{S^2} u(x, \alpha) v(\alpha)\, \mathrm{d}\alpha - 1 + \exp(-ik\theta \cdot x) \int_{S^2} (u_\delta - u) v\, \mathrm{d}\alpha \right\|_{a_{1,\varepsilon}}$$

$$\leqslant \mathscr{F}_\varepsilon(v) + \exp(k|\mathrm{Im}\,\theta|\varepsilon^{-1}) \int_{S^2} \|u_\delta - u\|_{a_{1,\varepsilon}} |v|\, \mathrm{d}\alpha. \tag{16}$$

One has

$$\|u_\delta - u\|_{a_{1,\varepsilon}} \leqslant \max_{\alpha \in S^2} \left\| \sum_{j=0}^{N} (A_{\delta j}(\alpha) - A_j(\alpha)) Y_j(\theta') h_j(kr) \right\|_{a_{1,\varepsilon}}$$

$$+ \max_{\alpha \in S^2} \left\| \sum_{j=N+1}^{\infty} A_j(\alpha) Y_j(\theta') h_j(kr) \right\|_{a_{1,\varepsilon}}$$

$$\leqslant \delta\lambda(N) + \omega(N) \tag{17}$$

where $\omega(N) \to 0$ as $N \to \infty$ and $\lambda(N) \to +\infty$ as $N \to \infty$. The function

$$\omega(N) := \max_{\alpha \in S^2} \left\| \sum_{j=N+1}^{\infty} A_j(\alpha) Y_j(\theta') h_j(kr) \right\|_{a_{1,\varepsilon}}$$

and the series converges for q satisfying (5), so that $\omega(N) \to 0$ as $N \to \infty$. We will prove that $\omega(N)$ and $\lambda(N)$ can be chosen independently of ε and θ (see lemma 1 below). The function

$$\lambda(N) \leqslant \lambda_1(N, \varepsilon) := \sum_{j=0}^{N} \delta^{-1} \max_{\alpha \in S^2} |A_{\delta j}(\alpha) - A_j(\alpha)| \, \| Y_j(\theta') h_j(kr) \|_{a_{1,\varepsilon}}$$

and $\lambda_1(N) \to +\infty$ as $N \to \infty$ because

$$\delta^{-1} \max_{\substack{\alpha \in S^2 \\ j \geqslant 0}} |A_{\delta j}(\alpha) - A_j(\alpha)| \leqslant c$$

$$\lambda(N) \leqslant c \sum_{j=0}^{N} \| Y_j(\theta') h_j(kr) \|_{a_{1,\varepsilon}}$$

and the series $\Sigma_{j=0}^{\infty} \| Y_j(\theta') h_j(kr) \|_{a_{1,\varepsilon}}$ diverges. Without loss of generality one can assume that $\lambda(N)$ is a continuous monotonically increasing-to-infinity function of N while $\omega(N)$ is a continuous monotonically decreasing-to-zero function of N. Let $N(\delta)$ be the solution to the problem

$$\delta\lambda(N) + \omega(N) = \min. \tag{18}$$

It is clear that $N(\delta) \to \infty$ as $\delta \to 0$, and that

$$\mu(\delta) := \delta\lambda(N(\delta)) + \omega(N(\delta)) \to 0 \qquad \text{as } \delta \to 0. \tag{19}$$

Explicit estimates for $\lambda(N)$ and $\omega(N)$ are given in Lemma 1 (see formulae (28) and (29) below) and we prove that $N(\delta)$ can be chosen independently of ε and θ as the solution to equation (18′) below.

Let

$$\int_{S^2} |v| \mathrm{d}\alpha := a(v). \tag{20}$$

It follows from (16), (17), (19) and (20) that

$$\mathscr{F}_{\delta\varepsilon}(v) \leqslant \mathscr{F}_{\varepsilon}(v) + \exp(k|\mathrm{Im}\,\theta|\varepsilon^{-1})a(v)\mu(\delta). \tag{21}$$

If one takes $v = v_\varepsilon(\alpha, \theta)$ in (21), where $v_\varepsilon(\alpha, \theta)$ is the sequence constructed above in (13), then

$$\mathscr{F}_{\delta\varepsilon}(v_\varepsilon) \leqslant \mathscr{F}_{\varepsilon}(v_\varepsilon) + \exp(k|\mathrm{Im}\,\theta|\varepsilon^{-1})a(v_\varepsilon)\mu(\delta) \leqslant d_\varepsilon + 1 + \exp(k|\mathrm{Im}\,\theta|\varepsilon^{-1})a(v_\varepsilon)\mu(\delta). \tag{22}$$

Since $\mu(\delta) \to 0$ as $\delta \to 0$ and $\mu(\delta)$ does not depend on ε and θ, one can find $\varepsilon = \varepsilon(\delta)$ such that $\varepsilon(\delta) \to 0$ as $\delta \to 0$ so slowly that

$$\exp(k|\mathrm{Im}\,\theta|\varepsilon^{-1}(\delta))a(v_{\varepsilon(\delta)})\mu(\delta) \leqslant c \qquad \delta \to 0 \tag{23}$$

where $c > 0$ is a constant. Then the sequence $v_{\varepsilon(\delta)} := v_\delta(\alpha, \theta)$ can be used for the stable recovery of $\tilde{q}(p)$ by the formula

$$\tilde{q}_\delta(p) = -4\pi \int_{S^2} \tilde{A}_\delta(\theta', \alpha) v_\delta(\alpha, \theta) \mathrm{d}\alpha, \qquad \theta - \theta' = p, \quad \theta', \theta \in M. \tag{24}$$

where

$$\tilde{A}_\delta(\theta', \alpha) = \sum_{j=0}^{N(\delta)} A_{\delta j}(\alpha) Y_j(\theta')$$

$N(\delta) \to \infty$ as $\delta \to 0$, and $N(\delta)$ solves (18), $\theta = \theta(\delta)$, $\theta' = \theta'(\varepsilon)$, $|\theta(\delta)| \xrightarrow[\delta\to 0]{} \infty$.

Indeed, it follows from (10) that

$$\tilde{q}(p) = -4\pi \int_{S^2} A(\theta', \alpha) v_\delta(\alpha, \theta) \mathrm{d}\alpha + m(\delta) \tag{25}$$

where $m(\delta) \to 0$ as $\delta \to 0$ and $\theta = \theta(\delta) \in M$ is chosen so that $|\theta(\delta)| \to \infty$, $\theta(\delta) \in M$, $\theta(\delta) - \theta'(\delta) = p$ and (23) holds with $\theta = \theta(\delta)$. Therefore

$$\begin{aligned} |\tilde{q}(p) - \tilde{q}_\delta(p)| &\leqslant m(\delta) + 4\pi \left| \int_{S^2} [\tilde{A}_\delta(\theta', \alpha) - A(\theta', \alpha)] v_\delta(\alpha, \theta) \mathrm{d}\alpha \right| \\ &\leqslant m(\delta) + cN^2(\delta)\delta a(v_\delta) b(\delta) + m_1(\delta) a(v_\delta) \end{aligned} \tag{26}$$

where $\theta, \theta' \in M$, $\theta' = \theta(\delta) - p$, and

$$b(\delta) := b(N(\delta), \theta(\delta)) = \max_{0 \leqslant j \leqslant N(\delta)} |Y_j(\theta')|$$

$$m_1(\delta) = c \left(\frac{a}{R}\right)^{N(\delta)} N^{1/2}(\delta) \exp(|\mathrm{Im}\,\theta(\delta)|R) \qquad R > a.$$

The technical results we need are formulated in the following lemma.

Lemma 1. Let $N(\delta)$ solve equation (18′) below. Let $R > 2a_1$, $a_1 > a$, $a_1 \geqslant 1$, $p \in \mathbb{R}^3$ be an arbitrary vector, $\theta(\delta) \in M$, $\theta'(\delta) \in M$, $\theta(\delta) - \theta'(\delta) = p$. One can choose $|\theta(\delta)| \to \infty$ and $\varepsilon(\delta) \to 0$ as $\delta \to 0$, such that (23) holds. Then

$$\lim_{\delta \to 0} a(v_\delta)[\delta N^2(\delta) b(\delta) + m_1(\delta)] = 0 \tag{27}$$

The following estimates hold for $\lambda(N)$ and $\omega(N)$:

$$\lambda(N) \leqslant c_1 \exp(-N) \left(\frac{N+1}{a_1}\right)^{N+1} \qquad N >> 1, \quad a_1 \geqslant a, \quad a_1 \geqslant 1, \tag{28}$$

$$\omega(N) \leqslant c_2 \left(\frac{a}{2a_1}\right)^N \qquad N >> 1, \quad a_1 > a, \quad a_1 \geqslant 1. \tag{29}$$

The constants $c_j > 0, j = 1, 2$, in (28) and (29) depend on a and on $\|q\|_{L^2(B_a)}$ and do not depend on δ, $\varepsilon(\delta)$ and $\theta(\delta)$. The solution $N(\delta)$ to the equation

$$\delta \exp(-N) \left(\frac{N+1}{a_1}\right)^{N+1} + c\left(\frac{a}{2a_1}\right)^N = \min \qquad a_1 > a \tag{18′}$$

is

$$N(\delta) = \frac{\ln(\delta^{-1})}{\ln\ln(\delta^{-1})}[1 + o(1)] \qquad \text{as } \delta \to 0.$$

Therefore one can recover $\tilde{q}(p)$ stably in the sense that a function $\tilde{q}_\delta(p)$, defined by formula (24), satisfies the inequality

$$|\tilde{q}(p) - \tilde{q}_\delta(p)| \leqslant \phi(\delta) \to 0 \qquad \text{as } \delta \to 0. \tag{30}$$

Formula (30) follows from (26), (27) and the relation $m(\delta) \to 0$ as $\delta \to 0$.

Claim. If $\tilde{q}_\delta(p)$ is known such that (30) holds then one can find $q_\delta(x)$ such that (7) holds.

Indeed, define

$$q_\delta(x) := \frac{1}{(2\pi)^3} \int_{|p| \leqslant R(\delta)} \exp(-ip \cdot x) \tilde{q}_\delta(p) \mathrm{d}p. \tag{31}$$

Then, for a suitable $R(\delta)$, $R(\delta) \to +\infty$ as $\delta \to 0$, one has (7). To prove this we use Parseval's equality:

$$\begin{aligned}\|q_\delta(x) - q(x)\|^2_{L^2(B_a)} &= \frac{1}{(2\pi)^3} \int |\tilde{q}_\delta(p) - \tilde{q}(p)|^2 \mathrm{d}p \\ &= \frac{1}{(2\pi)^3} \int_{|p| \leqslant R(\delta)} |\tilde{q}_\delta(p) - \tilde{q}(p)|^2 \mathrm{d}p + \frac{1}{(2\pi)^3} \int_{|p| \geqslant R(\delta)} |\tilde{q}|^2 \mathrm{d}p \\ &\leqslant \frac{1}{(2\pi)^3} \phi^2(\delta) \frac{4\pi}{3} R^3(\delta) + \frac{1}{(2\pi)^3} \psi(\delta)\end{aligned} \tag{32}$$

where

$$\psi(\delta) := \left(\frac{1}{2\pi}\right)^3 \int_{|p| \geqslant R(\delta)} |\tilde{q}(p)|^2 \mathrm{d}p \to 0 \qquad \text{as } R(\delta) \to \infty.$$

Therefore the right-hand side of (32) goes to zero provided that $R(\delta) \to \infty$ so that $\phi^2(\delta)R^3(\delta) \to 0$ as $\delta \to 0$. The claim is proved. Our argument establishes the following result.

Theorem 1. There exists a sequence $v_\delta(\alpha, \theta)$ such that formula (24) gives a stable approximation of $\tilde{q}(p)$ provided that $N(\delta)$ solves (18′), $\theta(\delta)$, $\theta'(\delta) \in M$, $\theta(\delta) - \theta'(\delta) = p \in \mathbb{R}^3$, $|\theta(\delta)| \to \infty$ and $\varepsilon(\delta) \to 0$ as $\delta \to 0$, and $\theta(\delta)$ and $\varepsilon(\delta)$ are chosen so that (23) holds.

Let us describe a numerical construction of the sequence $v_\delta(\alpha, \theta)$, the existence of which is established in theorem 1. Let $v_{\delta,\varepsilon,n}(\alpha, \theta)$ be a minimising sequence for the functional (15). It is possible to choose a member of the minimising sequence $n = n(\delta)$ and the functions $N(\delta)$, $\varepsilon = \varepsilon(\delta)$ and $\theta = \theta(\delta)$ so that

$$\mathscr{F}_{\delta\varepsilon}(v_{\delta,\varepsilon,n}) \leqslant c \tag{33}$$

where c = constant > 0 and c does not depend on δ. This is possible as was shown above. Let $v_\delta := v_{\delta,\varepsilon(\delta),n(\delta)}$ and let $\theta = \theta(\delta)$ be chosen as in theorem 1. This $v_\delta = v_\delta(\alpha, \theta)$ can be used in formula (24) for computing of $\tilde{q}_\delta(p)$ which gives a stable approximation of $\tilde{q}(p)$. In order to find $n(\delta)$, $N(\delta)$, $\varepsilon(\delta)$ and $\theta(\delta)$ numerically one takes a sequence of $\delta \to 0$ and, for each $\delta > 0$, solves problem (15) for a sequence of $\varepsilon \to 0$, $|\theta| \to \theta, \theta, \theta' \in M$, $\theta - \theta' = p$, and $N \to \infty$, and finds $N(\delta)$, $\varepsilon(\delta)$, $\theta(\delta)$ and $n(\delta)$ such that (33) and (23) hold. Here $n(\delta)$ is an integer identifying a member of a minimising sequence for the problem (15) as $\delta \to 0$.

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