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LETTER TO THE EDITOR

Necessary and sufficient condition on the fixed-energy scattering data for the potential to be spherically symmetric

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Abstract. Consider the Schrödinger equation for fixed energy and a real-valued compactly supported potential $q(x) \in L^2_{\text{loc}}$, $x \in \mathbb{R}^3$. Let $A(\theta', \theta)$ be the corresponding scattering amplitude for incident and exit directions θ and θ' respectively. It is proved that the potential is spherically symmetric if and only if $A(\theta', \theta)$ for this single energy depends only on the angle between θ and θ', that is $A(\theta', \theta) = A(\theta' \cdot \theta)$ for all $\theta', \theta \in S^2$, the unit sphere. Thus, if this condition holds at a single energy $\kappa^2 > 0$ it holds for all $\kappa > 0$ under the above assumption about $q(x)$.

Let

$$[\nabla^2 + \kappa^2 - q(x)]u = 0 \qquad \text{in } \mathbb{R}^3 \qquad \kappa > 0 \text{ is fixed.} \tag{1}$$

Without loss of generality let us assume in what follows that $\kappa = 1$. The scattering solution is defined as the solution to (1) with the asymptotics

$$u = \exp(\mathrm{i}\theta \cdot x) + A(\theta', \theta) r^{-1} \exp(\mathrm{i}r) + o(r^{-1}) \qquad \text{as } r = |x| \to \infty,\ xr^{-1} = \theta'. \tag{2}$$

The function $A(\theta', \theta)$ is the scattering amplitude. Let us assume

$$q \in Q_a := \{q\colon q(x) = \bar{q}(x) \in L^2(B_a),\ q(x) = 0 \text{ in } B'_a\} \tag{3}$$

where $B_a = \{x\colon |x| \leqslant a\}$, $B'_a := \mathbb{R}^3 \backslash B_a$. If $q(x) = q(r)$, $r = |x|$, then it is well known that

$$A(\theta', \theta) = A(\theta' \cdot \theta) \qquad \forall \theta', \theta \in S^2 \tag{4}$$

where S^2 is the unit sphere in $\mathbb{R}^3$. The basic result of this letter, theorem 1, has been formulated as a conjecture in [1].

Theorem 1. Assume that (3) holds. Then $q(x) = q(r)$ iff (4) holds. Thus (4) holds at all energies iff it holds at one.

The result of this type in the case when $A(\theta', \theta, \kappa) = A(\theta' \cdot \theta, \kappa)$ for all $\theta', \theta \in S^2$ and all $\kappa > 0$ has been proved in [1]. In [2–4] a theory for inverse scattering problems with fixed-energy data is given and in [5–6] one finds basic properties of the scattering amplitude. Our proof is very short and is based on the uniqueness theorem proved in [2–4, 7, 8].

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Proof. First we formulate the uniqueness theorem from [4].

Theorem 2. If $q_j \in Q_a$, $j = 1, 2$, and $A_1(\theta', \theta) = A_2(\theta', \theta)$ for all $\theta', \theta \in S^2$ then $q_1 = q_2$. Here $A_j(\theta', \theta)$ is the scattering amplitude corresponding to q_j, $j = 1, 2$, at a fixed $\kappa > 0$.

Assume that (4) holds. Our basic idea is to show that (4) implies

$$A_q(\theta', \theta) = A_{q \circ R}(\theta', \theta) \qquad \forall \theta', \theta \in S^2 \text{ and } \forall R \in \mathrm{O}(3) \tag{5}$$

where O(3) is the group of rotations in $\mathbb{R}^3$. Here $A_q(\theta', \theta)$ is the scattering amplitude corresponding to the potential $q(x)$, and $q \circ R$ is the potential whose value at a point x is $(q \circ R)(x) := q(R^{-1}x)$. Since $q(R^{-1}x) \in Q_a$ if $q(x) \in Q_a$, it follows from theorem 2 and equation (5) that

$$q(x) = q(R^{-1}x) \qquad \forall R \in \mathrm{O}(3). \tag{6}$$

This means that $q(x) = q(|x|) = q(r)$. Thus the proof is complete as long as (5) is established. Equation (5) follows from the equations

$$A_{q \circ R}(R\theta', R\theta) = A_q(\theta', \theta) \tag{7}$$

and

$$A_q(\theta', \theta) = A_q(R\theta', R\theta). \tag{8}$$

Equation (7) holds for any $q(x)$ and is an immediate consequence of the definition of the scattering amplitude $A_q(\theta', \theta)$. Equation (8) follows from our basic assumption (4), since $\theta' \cdot \theta = R\theta' \cdot R\theta$ for any $R \in \mathrm{O}(3)$.

Let us explain (7) in detail. Let $A_q(\theta', \theta)$ be the scattering amplitude in a certain coordinate system, say τ. Consider the same scattering amplitude in a coordinate system τ' in which any vector which in the coordinate system τ was x is equal to Rx, where $R \in \mathrm{O}(3)$. In the coordinate system τ' the potential q evaluated at a point y has the value $(q \circ R)(y) = q(R^{-1}y)$. For example, suppose that the potential q evaluated at a point x in the coordinate system τ has the value $q(x) = d \cdot x$, where d is a constant vector and the dot denotes inner product. Then, in the coordinate system τ', vector d is equal to Rd, so that the potential q evaluated at a vector y in the coordinate system τ' is equal to $Rd \cdot y = d \cdot R^{-1}y = q(R^{-1}y) := (q \circ R)(y)$. Here we use the fact that $R' = R^{-1}$, where R' is the transposed transformation; that is, the fact that rotations in $\mathbb{R}^3$ are orthogonal transformations. Thus, in the coordinate system τ' the value of the scattering amplitude $A_q(\theta', \theta)$ is equal to $A_{q \circ R}(R\theta', R\theta)$. This is the desired equation (7). Theorem 1 is proved.

Remark 1. Our argument shows that, for any class of potentials $q(x)$ for which the uniqueness theorem for the inverse scattering problem holds for certain data, condition (4) is necessary and sufficient for $q(x)$ to be spherically symmetric if this condition holds for the above data. For example, if (4) holds for all sufficiently large $\kappa > 0$ then $q(x) = q(|x|)$ if q belongs to the class of potentials for which the uniqueness of the solution of the inverse scattering problem is proved for the data $A(\theta', \theta, \kappa)$ given for all $\theta', \theta \in S^2$ and all κ sufficiently large. These are potentials $q(x)$ for which $|q(x)| \leq c(1 + |x|)^{-b}$, $b > 3$, $c = \text{constant} > 0$; see [6] (for an even larger class of potentials with $b > 1$, see [9]).

The argument in this paper is applicable to the scattering by obstacles such as, for example, in the problem considered in [1, 7, 10].

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