Electromagnetic inverse problems with surface measurements at low frequencies

Alexander G Ramm† and Erkki Somersalo‡

† Mathematics Department, Kansas State University, Manhattan, KS 66506, USA
‡ Department of Mathematics, University of Helsinki, Hallituskatu 15, 00100 Helsinki, Finland

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Abstract. This paper deals with the following inverse problem. A time-harmonic magnetic dipole is moved along the surface of the Earth; inside the Earth there is a bounded regular region of inhomogeneity where the electromagnetic parameters ε, μ and σ (permittivity, permeability and conductivity) vary; outside this region they are assumed to be constants. The scattered electric or magnetic field is then measured everywhere on the surface of the Earth with low transmitter frequencies. The problem is to find the parameters describing the inhomogeneity from these measurements. An exact analytic inversion formula is given in the special case when μ and σ are positive constants and ε is allowed to vary. The general case is treated in this work approximately by using a version of the Born inversion.

1. Introduction

In this work we consider the following electromagnetic inverse problem arising, for example, in geophysical exploration. An electromagnetic point source is moved along the surface of the Earth, modelled as a half space. The Earth is assumed to contain a bounded regular region of inhomogeneity where the electromagnetic parameters ε, μ and σ (permittivity, permeability and conductivity) vary; outside this region they are assumed to be constants. The scattered electric or magnetic field is then measured everywhere on the surface of the Earth with low transmitter frequencies. The problem is to find the parameters describing the inhomogeneity from these measurements.

We shall give a complete solution to the above problem in the special case when μ and σ are constants and ε is allowed to vary (section 3), whereas the general case is solved in this work approximately. In the special case an exact analytical solution is found. The key observation is that under these restrictions the low-frequency limit of the electromagnetic fields yields a linear integral equation from which ε is found. In the general case when also μ and σ vary the linearity is lost. However, the low-frequency asymptotics of the fields makes it possible to treat the general problem approximately. In section 4 we give an approximate method for solving the electromagnetic parameters from low-frequency data. Our method is a version of the Born inversion widely used in various inverse problems.

The general idea of using low-frequency limits for exact inversion from near-field measurements was used originally in [1] and [2] (see also [3, 4]). As a general reference, see [5].

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2. Low-frequency limits

We shall start by considering the low-frequency behaviour of the electromagnetic field. Assume that the time dependence of the field is harmonic, proportional to the factor $e^{-i\omega t}$, where ω is the transmitter frequency. Let the permeability μ and dielectric parameter $\varepsilon' = \varepsilon + i\sigma/\omega$ be in $C^2(\mathbb{R}^3)$, i.e. μ and ε' are twice continuously differentiable, and assume that $\mu > 0$ and $\varepsilon > 0$. Regarding the conductivity σ, it is assumed that either $\sigma > 0$ everywhere or σ vanishes identically. Furthermore, we shall assume that there are constants μ_0 and $\varepsilon'_0 = \varepsilon_0 + i\sigma_0/\omega$ and a bounded regular domain $G \subset \mathbb{R}^3_- = \{(x_1, x_2, x_3) \in \mathbb{R} \mid x_3 < 0\}$ such that $\varepsilon' = \varepsilon'_0$ and $\mu = \mu_0$ outside G. The complex amplitudes of the electric and magnetic fields (E, H) satisfy Maxwell's equations

$$
\begin{aligned}
&\nabla \times H + i\omega\epsilon' E = j \\
&\nabla \times E - i\omega\mu H = 0 \\
&\nabla \cdot (\epsilon' E) = \nabla \cdot j / i\omega \\
&\nabla \cdot (\mu H) = 0.
\end{aligned}
\tag{2.1}
$$

Here j is the external current. We shall consider the case when j corresponds to a magnetic dipole at y outside G pointing towards the positive x_3 direction. Explicitly j is obtained by shrinking a current loop in the (x_1, x_2) plane to a single point keeping the magnetic moment of the loop fixed. If the current density of the loop with radius $a > 0$ and current I is in cylindrical coordinates $x - y = (\rho, \varphi, z)$ given by

$$
j_a(x, y) = I\delta(\rho - a)\delta(z)e_\varphi
$$

e_φ being the azimuthal unit vector, then it is easy to verify with Stokes' formula that letting a go to zero with $\pi a^2 I = 1$, j_a approaches in the distributional sense the density

$$
j(x, y) = \nabla \times (\delta(x - y)e_3)
$$

where e_3 is the unit vector along the x_3 axis. Note that especially $\nabla \cdot j = 0$. To specify the solution of the system (2.1) uniquely, we shall assume that the solutions E and H satisfy the radiation condition at infinity. Denoting $k^2 = \omega^2\mu_0\varepsilon'_0$, $0 \le \arg k < \pi$, we assume that for $\omega > 0$

$$
\begin{aligned}
&\omega\mu_0 x^0 \times H + kE = O\left(\frac{1}{|x|^2}\right) \\
&\omega\varepsilon'_0 x^0 \times E - kH = O\left(\frac{1}{|x|^2}\right)
\end{aligned}
\tag{2.2}
$$

where $x^0 = x/|x|$. It is well known that equations (2.2) imply

$$
|E| = O\left(\frac{1}{|x|}\right) \qquad |H| = O\left(\frac{1}{|x|}\right) \tag{2.3}
$$

as $|x| \to \infty$.

To get an integral representation for the fields we write $E = E_i + E_{sc}$, $H = H_i + H_{sc}$, where the incident fields E_i and H_i satisfy equations (2.1) with $\varepsilon' = \varepsilon_0'$ and $\mu' = \mu_0$. Explicitly,

$$\begin{aligned} E_i(x, y, \omega) &= -i\omega\mu_0 \nabla \times (e_3 g(x-y, \omega)) \\ H_i(x, y, \omega) &= \nabla \times (\nabla \times (e_3 g(x-y, \omega))) \qquad x \neq y \end{aligned} \tag{2.4}$$

where $\nabla = \nabla_x$ and

$$g(x, \omega) = \frac{e^{ik|x|}}{4\pi|x|}.$$

Writing $\tilde{\mu} = \mu - \mu_0$, $\tilde{\varepsilon}' = \varepsilon' - \varepsilon_0'$, equations (2.1) give

$$\begin{aligned} \nabla \times (\nabla \times E_{sc}) &= \nabla \times (\nabla \times E) - \nabla \times (\nabla \times E_i) \\ &= i\omega \nabla \times (\tilde{\mu} H) + i\omega\mu_0 \nabla \times H - i\omega\mu_0 \nabla \times H_i \\ &= i\omega \nabla \times (\tilde{\mu} H) + \omega^2 \mu_0 \tilde{\varepsilon}' E + k^2 E_{sc}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \nabla \times (\nabla \times E_{sc}) &= \nabla(\nabla \cdot E_{sc}) - \Delta E_{sc} \\ &= -\nabla \left(\frac{\nabla \varepsilon'}{\varepsilon'} \cdot E \right) - \Delta E_{sc}. \end{aligned}$$

Therefore,

$$(\Delta + k^2) E_{sc} = -\nabla \left(\frac{\nabla \varepsilon'}{\varepsilon'} \cdot E \right) - \omega^2 \mu_0 \tilde{\varepsilon}' E - i\omega \nabla \times (\tilde{\mu} H)$$

and after partial integrations E is found to satisfy

$$\begin{aligned} E(x, y, \omega) &= E_i(x, y, \omega) + E_{sc}(x, y, \omega) \\ &= E_i(x, y, \omega) + \int_G \bigg(-\frac{\nabla \varepsilon'(z, \omega)}{\varepsilon'(z, \omega)} \cdot E(z, y, \omega) \nabla g(x-z, \omega) \\ &\quad + \omega^2 \mu_0 \tilde{\varepsilon}'(z, \omega) E(z, y, \omega) g(x-z, \omega) + i\omega \tilde{\mu}(z) H(z, y, \omega) \times \nabla g(x-z, \omega) \bigg) dz \end{aligned} \tag{2.5a}$$

where the differentiations are taken with respect to z. Similarly,

$$\begin{aligned} H(x, y, \omega) &= H_i(x, y, \omega) + \int_G \bigg(-\frac{\nabla \mu(z)}{\mu(z)} \cdot H(z, y, \omega) \nabla g(x-z, \omega) \\ &\quad + \omega^2 \varepsilon_0' \tilde{\mu}(z) H(z, y, \omega) g(x-z, \omega) - i\omega \tilde{\varepsilon}'(z, \omega) E(z, y, \omega) \times \nabla g(x-z, \omega) \bigg) dz. \end{aligned} \tag{2.5b}$$

It is known (see [6]) that equations (2.5*a*, *b*) have a unique solution (E, H), which is continuous outside $\{y\}$ and that this pair of equations is equivalent to the system (2.1) with radiation condition.

The solution to the class of inverse problems under consideration is based on the low-frequency behaviour of the electromagnetic fields. The proof of the following proposition follows the lines of [7] (see also section V.4 of [5]).

Proposition 2.1. The solution (E, H) of equations (2.5*a*, *b*) has a low-frequency limit

$$\lim_{\omega\to 0}(E(x,y,\omega), H(x,y,\omega)) = (0, H_0(x,y))$$

in $H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{y\})$, and H_0 satisfies the equations

$$\nabla \times H_0 = j \qquad \nabla \cdot (\mu H_0) = 0 \tag{2.6}$$

with the asymptotic behaviour

$$|H_0| = O\left(\frac{1}{|x|}\right).$$

Proof. Obviously, as ω tends to zero, $(E_{\mathrm{i}}, H_{\mathrm{i}})$ converges to $(0, H_{\mathrm{i},0})$ in $H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{y\})$, where

$$H_{\mathrm{i},0}(x,y) = \nabla \times (\nabla \times (e_3 g_0(x-y)))$$

and

$$g_0(x) = \frac{1}{4\pi|x|}.$$

Hence, we have to prove the convergence for the scattered field $(E_{\mathrm{sc}}, H_{\mathrm{sc}})$. Let us denote by L^2_δ, $\delta \in \mathbb{R}$, the weighted L^2 space

$$L^2_\delta = \left\{ f \in L^2_{\mathrm{loc}} \mid \|f\|_\delta = \left(\int |f(x)|^2(1+|x|^2)^{-\delta}\,\mathrm{d}x\right)^{1/2} < \infty \right\} \qquad \int := \int_{\mathbb{R}^3}.$$

Since the scattered field satisfies the estimate (2.3), we have

$$N(\omega) = (\|E_{\mathrm{sc}}\|^2_\delta + \|H_{\mathrm{sc}}\|^2_\delta)^{1/2} < \infty$$

for each $\omega > 0$ and $\delta > 1$ fixed. Here we have suppressed the y dependence of the fields. Assume for a moment that

$$\sup_{0<\omega\le\omega_0} N(\omega)^2 < \infty \tag{2.8}$$

for some $\omega_0 > 0$. We shall show that this already implies proposition 2.1.

Let $B_R = \{x \mid |x| \le R\}$ with $R > 0$ arbitrary. Since $(E_{\mathrm{sc}}, H_{\mathrm{sc}})$ satisfies the system

$$\begin{aligned} \nabla \times H_{\mathrm{sc}} &= -\mathrm{i}\omega\varepsilon' E_{\mathrm{sc}} - \mathrm{i}\omega\tilde{\varepsilon}' E_{\mathrm{i}} \\ \nabla \times E_{\mathrm{sc}} &= \mathrm{i}\omega\mu H_{\mathrm{sc}} + \mathrm{i}\omega\tilde{\mu} H_{\mathrm{i}} \\ \nabla \cdot (\varepsilon' E_{\mathrm{sc}}) &= -\nabla \cdot (\tilde{\varepsilon}' E_{\mathrm{i}}) \\ \nabla \cdot (\mu H_{\mathrm{sc}}) &= -\nabla \cdot (\tilde{\mu} H_{\mathrm{i}}) \end{aligned} \tag{2.9}$$

as can be easily checked, we have by (2.8)

$$\int_{B_R} (|\nabla \times E_{\mathrm{sc}}(x,\omega)|^2 + |\nabla \times H_{\mathrm{sc}}(x,\omega)|^2)\,\mathrm{d}x \le C(R)$$

and

$$\int_{B_R} (|\nabla \cdot E_{sc}(x,\omega)|^2 + |\nabla \cdot H_{sc}(x,\omega)|^2)\,dx \le C(R)$$

for some $C(R)$ independent of ω, $0 < \omega \le \omega_0$. These estimates imply that E_{sc} and H_{sc} are in the Sobolev space $H^1(B_R)$ with norms bounded uniformly with respect to ω, $0 < \omega \le \omega_0$. Indeed, if $\varphi \in C^\infty$, supp $\varphi \subset B_{R+1}$, $\varphi|_{B_R} = 1$, we have $\nabla \times (\varphi E_{sc}) \in L^2(B_{R+1})$, $\nabla \cdot (\varphi E_{sc}) \in L^2(B_{R+1})$, and using the well known representation of a vector field in terms of its divergence and curl we get for $x \in B_R$

$$\begin{aligned} E_{sc}(x,\omega) &= \varphi(x)E_{sc}(x,\omega) \\ &= \int \nabla \cdot (\varphi(z)E_{sc}(z,\omega))\nabla g_0(x-z)\,dx \\ &\quad + \int \nabla \times (\varphi(z)E_{sc}(z,\omega)) \times \nabla g_0(x-z)\,dz \end{aligned} \tag{2.10}$$

where the integral operators are smoothing by one index. The H_{sc} field is treated similarly.

Hence by the compact embedding theorem for Sobolev spaces we can find a sequence $\omega_n \to 0$ such that

$$(E_{sc}(x,\omega_n), H_{sc}(x,\omega_n)) \to (E_0(x), H_0(x)) \tag{2.11}$$

in L^2_{loc}. By (2.9) and (2.10) the convergence takes place also in H^2_{loc}, and the limit (E_0, H_0) satisfies (2.9) with $\omega = 0$, i.e.

$$\begin{aligned} &\nabla \times H_0 = \sigma E_0 \\ &\nabla \times E_0 = 0 \\ &\nabla \cdot (\gamma E_0) = 0 \\ &\nabla \cdot (\mu H_0) = -\nabla \cdot (\tilde{\mu} H_{i,0}). \end{aligned} \tag{2.12}$$

Here $\gamma = \sigma$ if $\sigma \neq 0$ and $\gamma = \varepsilon$ if $\sigma = 0$ identically. Note that by the assumptions, γ is strictly positive in both cases.

We shall show first that $E_0 = 0$. Since $\nabla \times E_0 = 0$, E_0 can be written as $E_0 = \nabla\varphi$, and $\nabla \cdot (\gamma \nabla \varphi) = 0$. By the integral equation (2.5*a*) it is clear that $\varphi = O(1/|x|)$. Since φ is harmonic near infinity, we have by Green's representation formula that $|\nabla\varphi| = O(1/|x|^2)$. Therefore,

$$\begin{aligned} \int_{B_R} \gamma(x)|\nabla\varphi(x)|^2\,dx &= \int_{B_R} \nabla \cdot (\overline{\varphi(x)}\gamma(x)\nabla\varphi(x))\,dx \\ &= \int_{\partial B_R} \overline{\varphi(x)}\gamma(x)\frac{\partial\varphi}{\partial\nu}(x)\,dS(x) \end{aligned}$$

by the divergence theorem. Here $\partial\varphi/\partial\nu$ denotes the outer normal derivative at the boundary. As $R \to \infty$, the right-hand side vanishes implying $E_0 = \nabla\varphi = 0$.

To show that the limit H_0 is independent of the choice of the sequence (ω_n) it is enough to notice that if H_0 satisfies

$$\nabla \times H_0 = 0 \qquad \nabla \cdot (\mu H_0) = 0$$

with the asymptotic behaviour (2.3) at infinity, the same argument as above implies that $H_0 = 0$.

Finally, we have to prove the estimate (2.8). Assuming the contrary that, for some $\omega_n \to 0$, $N(\omega_n) > n$ for each n, define

$$(E_n(x), H_n(x)) = \frac{1}{N(\omega_n)}(E_{\rm sc}(x, \omega_n), H_{\rm sc}(x, \omega_n))$$

whence

$$N_n = (\|E_n\|_\delta^2 + \|H_n\|_\delta^2)^{1/2} = 1. \tag{2.14}$$

By the above argument, $(E_n, H_n) \to (\tilde{E}, \tilde{H})$ in $H^2_{\rm loc}$ and $(\tilde{E}, \tilde{H})$ solves the system (2.12) with $(E_{{\rm i},0}, H_{{\rm i},0}) = 0$, implying that $(\tilde{E}, \tilde{H}) = 0$. On the other hand, if $R > 0$ is chosen large enough, Green's formula gives

$$E_n(x) = \int_{\partial B_R} \left(E_n(z) \frac{\partial g}{\partial \nu(z)}(x - z, \omega_n) - g(x - z, \omega_n) \frac{\partial E_n}{\partial \nu(z)}(z) \right) \, {\rm d}S(z)$$

and similarly for $H_n(x)$. Due to the convergence of E_n in $H^2_{\rm loc}$ the above formula implies that

$$|E_n(x)| \leq \frac{c}{|x|}$$

for some $c > 0$ independent of n, as $|x| > R + 1$. Therefore we have

$$\lim_{n \to \infty} N_n^2 = 0$$

contradicting (2.14). The proposition is thus proved. □

3. Exact inversion: a special case

In this section we shall consider the general low-frequency inverse problem in a restricted class of functions ε' and μ. We shall prove the following result.

Theorem 3.1. Assume that $\mu = \mu_0 > 0$ and $\sigma = \sigma_0 > 0$ everywhere in $\mathbb{R}^3$. Then the inhomogenity $\tilde{\varepsilon}(x) = \varepsilon(x) - \varepsilon_0$, $x \in G$, can be analytically computed from the low-frequency data $\{H(x, y, \omega) \mid 0 < \omega \leq \omega_0, x, y \in P, x \neq y\}$, where $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$.

The proof is based on the following application of the low-frequency limit proved in the previous section.

Lemma 3.2. Under the assumptions on the functions μ and ε' given above,

$$\lim_{\omega\to 0}\frac{1}{\omega^2}(H(x,y,\omega)-H_{\rm i}(x,y,\omega)) = -\mu_0\int \tilde{\varepsilon}(z)\nabla\times(e_3 g_0(z-y))\times\nabla g_0(x-z)\,{\rm d}z$$

the convergence taking place in $H^2_{\rm loc}$.

Proof of lemma 3.2. Denoting by K the dyadic kernel

$$K(x,z,\omega)=\nabla g(x-z,\omega)\frac{\nabla\varepsilon(z)}{\omega\varepsilon(z)+{\rm i}\sigma_0}-\omega\tilde{\varepsilon}(z)g(x-z,\omega)I$$

$0<\omega<\omega_0$ and I being the unit matrix in $\mathbb{R}^3$, E satisfies the integral equation

$$\begin{aligned}E(x,y,\omega) &= E_{\rm i}(x,y,\omega)-\omega\int_G K(x,z,\omega)E(z,y,\omega)\,{\rm d}z\\ &:= E_{\rm i}(x,y,\omega)+\omega\mathscr{K}E(x,y,\omega).\end{aligned}$$

Obviously, $\mathscr{K}$ is a mapping from $L^2_{\rm loc}$ to $H^1_{\rm loc}$ with norm locally bounded uniformly with respect to ω, $0<\omega<\omega_0$. Therefore,

$$\begin{aligned}\lim_{\omega\to 0}\frac{1}{\omega}E(x,y,\omega) &= \lim_{\omega\to 0}\frac{1}{\omega}E_{\rm i}(x,y,\omega)\\ &= -{\rm i}\mu_0\nabla\times(e_3 g_0(x-y))\end{aligned}$$

in $H^1_{\rm loc}$. Since H satisfies the equation

$$H(x,y,\omega)=H_{\rm i}(x,y,\omega)-{\rm i}\omega\int_G\tilde{\varepsilon}(z)E(z,y,\omega)\times\nabla g(x-z,\omega)\,{\rm d}z$$

the claim follows immediately, since the operator from $H^1_{\rm loc}$ to $H^2_{\rm loc}$ with the kernel $\nabla g(x-z,\omega)\times I$ tends to the one with $\nabla g_0(x-z)\times I$ as $\omega\to 0$. □

Proof of theorem 3.1. By the previous lemma it suffices to show that $\tilde{\varepsilon}$ can be uniquely and analytically recovered from the integral

$$I(x,y)=\int_G\tilde{\varepsilon}(z)\nabla\times(e_3 g_0(z-y))\times\nabla g_0(x-z)\,{\rm d}z \tag{3.1}$$

as $x,y\in P$. We shall denote $x=(x',0)$, $y=(y',0)$ and $z=(z',z_3)$, where $x',y',z'\in\mathbb{R}^2$. Using the identity

$$\int_{\mathbb{R}^2}\exp({\rm i}x'\cdot\xi)\frac{1}{|x-z|}\,{\rm d}x'=\frac{2\pi}{|\xi|}\exp({\rm i}\xi\cdot z')\exp(-|\xi||z_3|) \tag{3.2}$$

$\xi\in\mathbb{R}^2$ (see e.g. [5, p 220] or [8, p 91]), we see that the Fourier transform of I along $P\times P$ can be written as

$$\begin{aligned}\hat{I}(\xi,\eta) &\equiv \int_{\mathbb{R}^2}{\rm d}x'\int_{\mathbb{R}^2}{\rm d}y'\,\exp[{\rm i}(x'\cdot\xi+y'\cdot\eta)]I((x',0),(y',0))\\ &= -\frac{1}{4}({\rm i}\eta^0+(\eta^0\cdot\xi^0)e_3)\int_G\tilde{\varepsilon}(z)\exp[{\rm i}(\xi+\eta)\cdot z']\exp[(|\xi|+|\eta|)z_3]\,{\rm d}z\end{aligned}$$

where $\eta^0 = \eta/|\eta|$, $\xi^0 = \xi/|\xi|$.

We introduce new variables in this integral. Let $p = \xi + \eta \in \mathbb{R}^2$, $r = |\xi| + |\eta| \in \mathbb{R}_+$. Considering η^0 as a fixed parameter, the change of variables $(|\eta|, \xi) \to (r, p)$ is non-singular, since the Jacobian is

$$\left|\frac{\partial(r,p)}{\partial(|\eta|,\xi)}\right| = 1 - \xi^0 \cdot \eta^0$$
$$\neq 0 \qquad \text{unless } \xi^0 = \eta^0.$$

Indeed, we have the inverse mapping

$$|\eta| = r - |p - r\eta^0|^2(2(r - p\cdot\eta^0))^{-1}$$
$$\xi = p - r\eta^0 + \eta^0|p - r\eta^0|^2(2(r - p\cdot\eta^0))^{-1}$$

as $\xi^0 \neq \eta^0$. In terms of (p, r) we have

$$\frac{2}{\eta^0\cdot\xi^0}(\mathrm{i}\xi^0 - e_3)\cdot\hat{I} = \int_G \tilde{\varepsilon}(z)\exp(\mathrm{i}p\cdot z' + rx_3)\,\mathrm{d}z.$$

Since this integral is known for all $p \in \mathbb{R}^2$, $r \in \mathbb{R}_+$ and $\eta^0 \in S^1$ can be an arbitrary unit vector, the function $\tilde{\varepsilon}$ can be recovered analytically in a unique way. □

4. Approximate solution in the general case

It is easy to see why the procedure of the previous section fails to work in the general case: the low-frequency limits of the E and H fields depend in general on the unknown functions μ and σ, thus yielding a non-linear integral equation instead of the linear one (3.1). To get an approximate but linear equation for solving the functions ε' and μ in the general case one can try to use a version of the Born approximation as is often done in various inverse problems.

We shall start with equation (2.5*b*). Rearranging the terms according to the asymptotic behaviour at low frequencies we have

$$\begin{aligned} H(x,y,\omega) = {} & H_{\mathrm{i}}(x,y,\omega) - \int_G \frac{\nabla\mu(z)}{\mu(z)}\cdot H(z,y,\omega)\nabla g(x-z,\omega)\,\mathrm{d}z \\ & + \mathrm{i}\omega\left(\sigma_0\int_G \tilde{\mu}(z)H(z,y,\omega)g(x-z,\omega)\,\mathrm{d}z\right. \\ & \left. + \int_G \tilde{\sigma}(z)\frac{E(z,y,\omega)}{\mathrm{i}\omega}\times\nabla g(x-z,\omega)\,\mathrm{d}z\right) \\ & + \omega^2\left(\varepsilon_0\int_G \tilde{\mu}(z)H(z,y,\omega)g(x-z,\omega)\,\mathrm{d}z\right. \\ & \left. + \int_G \tilde{\varepsilon}(z)\frac{E(z,y,\omega)}{\mathrm{i}\omega}\times\nabla g(x-z,\omega)\,\mathrm{d}z\right) \\ = {} & H_{\mathrm{i}}(x,y,\omega) - I_0(x,y,\omega) + \mathrm{i}\omega I_1(x,y,\omega) + \omega^2 I_2(x,y,\omega). \end{aligned} \tag{4.1}$$

As ω tends to zero, the integrals have finite limits in H^2_{loc}. The proof is similar to that of lemma 3.2. The approximate linearisation is now done by using the Born approximations

$$\begin{aligned}\lim_{\omega\to 0} H(x,y,\omega) &\simeq \lim_{\omega\to 0} H_{\mathrm{i}}(x,y,\omega) = \nabla\times(\nabla\times(e_3 g_0(x-y)))\\ \lim_{\omega\to 0}\frac{1}{\mathrm{i}\omega}E(x,y,\omega) &\simeq \lim_{\omega\to 0}\frac{1}{\mathrm{i}\omega}E_{\mathrm{i}}(x,y,\omega) = -\mu_0\nabla\times(e_3 g_0(x-y))\end{aligned} \tag{4.2}$$

in H^2_{loc}. These approximations are justified if $\nabla\sigma/\sigma$, $\nabla\mu/\mu$ as well as $\tilde{\varepsilon}$, $\tilde{\mu}$ and $\tilde{\sigma}$ are small.

Using these approximations, we get approximate formulae for the Taylor coefficients of $H_{\mathrm{sc}}(x,y,\omega)$ at $\omega=0$, namely

$$\begin{aligned}\lim_{\omega\to o} I_0(x,y,\omega) &\simeq I_0^{\mathrm{B}}(x,y)\\ &:= \int_G \frac{\nabla\mu(z)}{\mu(z)}\cdot\nabla\times(\nabla\times(e_3 g_0(z-y)))\nabla g_0(x-z)\,\mathrm{d}z\end{aligned} \tag{4.3a}$$

$$\begin{aligned}\lim_{\omega\to o} I_1(x,y,\omega) &\simeq I_1^{\mathrm{B}}(x,y)\\ &:= \sigma_0\int_G \tilde{\mu}(z)\nabla\times(\nabla\times(e_3 g_0(z-y)))g(x-z)\,\mathrm{d}z\\ &\quad -\mu_0\int_G \tilde{\sigma}(z)(\nabla\times(e_3 g_0(z-y)))\times\nabla g_0(x-z)\,\mathrm{d}z\end{aligned} \tag{4.3b}$$

and

$$\begin{aligned}\lim_{\omega\to o} I_2(x,y,\omega) &\simeq I_2^{\mathrm{B}}(x,y)\\ &:= \varepsilon_0\int_G \tilde{\mu}(z)\nabla\times(\nabla\times(e_3 g_0(z-y)))g(x-z)\,\mathrm{d}z\\ &\quad -\mu_0\int_G \tilde{\varepsilon}(z)(\nabla\times(e_3 g_0(z-y)))\times\nabla g_0(x-z)\,\mathrm{d}z.\end{aligned} \tag{4.3c}$$

Theorem 4.1. The Born approximations $I_j^{\mathrm{B}}(x,y)$, $j=0,1,2$, $x,y\in P$, of the Taylor coefficients of $H_{\mathrm{sc}}(x,y,\omega)$ at $\omega=0$ determine the functions $\tilde{\varepsilon},\tilde{\sigma}$ and $\tilde{\mu}$ uniquely.

Proof. As in the proof of theorem 3.1, a straightforward application of formula (3.2) gives for the Fourier transform $(x',y')\to(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$ of $I_2^{\mathrm{B}}(x,y)$

$$\begin{aligned}\hat{I}_2^{\mathrm{B}}(\xi,\eta) &= \varepsilon_0\frac{|\eta|}{4|\xi|}(\mathrm{i}\eta^0+e_3)\int_G\tilde{\mu}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z\\ &\quad +\mu_0\frac{1}{4}(\mathrm{i}\eta^0+(\xi^0\cdot\eta^0)e_3)\int_G\tilde{\varepsilon}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z.\end{aligned}$$

Since $(\mathrm{i}\eta+|\eta|e_3)\cdot(\mathrm{i}\eta+|\eta|e_3)=0$ and $(\mathrm{i}\xi^0+e_3)\cdot(\mathrm{i}\eta^0+(\eta^0\cdot\xi^0)e_3)=0$, by choosing ξ^0 and η^0 so that $\xi^0\neq\eta^0$ we get

$$\frac{4|\xi|}{\varepsilon_0|\eta|(1-\xi^0\cdot\eta^0)}(\mathrm{i}\xi^0+e_3)\cdot\hat{I}_2^{\mathrm{B}}(\xi,\eta) = \int_G\tilde{\mu}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z$$

and

$$-\frac{4|\xi|}{\mu_0(1-\xi^0\cdot\eta^0)}(\mathrm{i}\eta^0+e_3)\cdot\hat{I}_2^{\mathrm{B}}(\xi,\eta)=\int_G\tilde{\varepsilon}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z.$$

The same change of variables as in the proof of theorem 3.1 shows that $\tilde{\mu}$ and $\tilde{\varepsilon}$ can be retrieved from $\hat{I}_2^{\mathrm{B}}(\xi,\eta)$ uniquely.

Similarly, we have

$$\hat{I}_1^{\mathrm{B}}(\xi,\eta)=\sigma_0\frac{|\eta|}{4|\xi|}(\mathrm{i}\eta^0+e_3)\int_G\tilde{\mu}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z$$
$$+\mu_0\frac{1}{4}(\mathrm{i}\eta^0+(\xi^0\cdot\eta^0)e_3)\int_G\tilde{\sigma}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z.$$

Since the first integral is now known, we get

$$\frac{4}{\mu_0}\mathrm{i}\eta^0\cdot\left(\hat{I}_1^{\mathrm{B}}(\xi,\eta)-\sigma_0\frac{|\eta|}{4|\xi|}(\mathrm{i}\eta^0+e_3)\int_G\tilde{\mu}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z\right)$$
$$=\int_G\tilde{\sigma}(z)\exp[\mathrm{i}(\xi+\eta)\cdot z'+(|\xi|+|\eta|)z_3]\,\mathrm{d}z$$

and $\tilde{\sigma}$ is finally recovered. □

Some general remarks about the validity of the Born approximation for solving inverse problems are given in [9] and in the appendix of [10].

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