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# Recovery of the potential from fixed-energy scattering data

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**Abstract.** It is proved that the scattering amplitude $A(\theta', \theta, k)$ known at a fixed $k$ for all $\theta$, $\theta' \in S^2$ determines a compactly supported $q(x)$ uniquely. A numerical procedure for recovery of the potential is given.

## 1. Introduction

Let $q(x) \in L^2(D)$, $D \subset R^3$ is a bounded domain with the smooth boundary $\Gamma$, $q$ is real-valued, $q = 0$ outside $D$. Let

$$lu := [\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } R^3 \tag{1}$$

$$u = \exp(\mathrm{i}k\theta \cdot x) + v \tag{2}$$

$$v = A(\theta', \theta, k) r^{-1} \exp(\mathrm{i}kr) + \mathrm{o}(r^{-1}) \qquad r = |x| \to \infty, \quad r^{-1}x = \theta'. \tag{3}$$

The inverse problem is to find $q(x)$ given $A(\theta', \theta, k)$ for all $\theta'$, $\theta \in S^2$ and a fixed $k > 0$. Uniqueness of the solution to this problem was announced in [1], here a detailed proof is given. Some related results are in [2–6].

## 2. Proof

*2.1.*

The starting point is the known formula

$$-4\pi A(\theta', \theta, k) = \int \exp(-\mathrm{i}k\theta' \cdot x) q(x) u(x, \theta, k)\, \mathrm{d}x. \tag{4}$$

The proof consists of the following steps.

*Step 1.* If $A(\theta', \theta, k)$ is known then the integrals

$$\int \exp(-\mathrm{i}k\theta' \cdot x) q(x) \psi(x, k)\, \mathrm{d}x \tag{5}$$

are known for any $H^2$ solution of the equation (1) in $D$, where $H^2$ is the Sobolev space $W^{2,2}(D)$.

*Step 2.* There exists a solution to (1) of the form

$$\psi = \exp(\mathrm{i}k\theta \cdot x)(1 + R) \qquad \theta \cdot \theta = 1, \quad \theta \in \mathbb{C}^3 \tag{6}$$

$$\|R\|_{L^\infty(D_1)} \to 0 \quad \text{as} \quad |\theta| \to \infty, \quad \theta \cdot \theta = 1, \quad \theta \in \mathbb{C}^3 \tag{7}$$

where $D_1 \subset R^3$ is an arbitrary bounded domain.

*Step 3.* Substitute (6) into (5) and let

$$|\theta| \to \infty, \quad |\theta'| \to \infty, \quad \theta \cdot \theta = \theta' \cdot \theta' = 1, \quad \theta, \theta' \in \mathbb{C}^3, \quad k(\theta - \theta') = p. \tag{8}$$

Then (5) becomes

$$\tilde{q}(p) = \int \exp(\mathrm{i}p \cdot x) q(x)\, \mathrm{d}x \tag{9}$$

so that $\tilde{q}(p)$ is found. Therefore $q(x)$ is recovered.

*Step 4.* Suppose there are two potentials $q_j(x)$, $j = 1, 2$ with the same scattering amplitude at a fixed $k > 0$. Then $\tilde{q}_1(p) = \tilde{q}_2(p)$, so that $q_1 = q_2$. Steps 2 and 3 were discussed in full detail in [1–3]. Let us discuss steps 1 and 4.

*2.2.*

*Step 1.* Let us prove that the closure of the linear span of the set $\{u(x, \theta', k)\}_{\forall\theta \in S^2}$ in $L^2(D)$ contains $N_D(l)$:

$$\mathcal{M} := \text{cl span}\{u(x, \theta, k)\}_{\forall\theta\in S^2} \supset N_D(l) \tag{10}$$

where $N_D(l) = \{\psi : l\psi = 0 \text{ in } D,\ \psi \in H^2\}$.

Without loss of generality assume that $k^2 \notin \sigma(l_D)$, that is $k^2$ is not an eigenvalue of the Dirichlet operator $l$ in $D$. If $k^2 \in \sigma(l_D)$ then take $D_1$ which contains $D$ and such that $k^2 \notin \sigma(l_{D_1})$. Since supp $q \subset D \subset D_1$ it is sufficient to recover $q$ in $D_1$. If $k^2 \notin \sigma(l_D)$ then the set span$\{u\}$ is dense in $L^2(D)$ in the set $N_D(l)$ if the set span $\{u(s, k)\}$, $s \in \Gamma$, is dense in $L^2(\Gamma)$. Indeed, if $m \in N_D(l)$, $k^2 \notin \sigma(l_D)$ and if $\psi \in N_D(l)$ and $\|\psi - m\|_{L^2(\Gamma)} \leqslant \varepsilon$ then $\|\psi - m\|_{L^2(D)} \leqslant c\varepsilon$, where $c$ is a constant that does not depend on $\varepsilon$. Here $m$ is any element of $\mathcal{M}$.

Suppose that there is an $\eta \in L^2(\Gamma)$ such that

$$\int_\Gamma \eta u \, \mathrm{d}s = 0 \qquad \forall \theta \in S^2. \tag{11}$$

Let us prove that (11) implies that $\eta = 0$. Define

$$w(x) := \int_\Gamma G(x, s)\eta \, \mathrm{d}s \tag{12}$$

where $G$ is the resolvent kernel of $l$:

$$lG = -\delta(x - y) \quad \text{in } R^3 \tag{13}$$

$$r\left(\frac{\partial G}{\partial r} - \mathrm{i}kr\right) \to 0 \qquad r = |x| \to \infty. \tag{14}$$

It is known (see [4], p 46) that

$$G(x, s, k) = \frac{\exp(\mathrm{i}kr)}{r} u(s, \theta, k)(1 + \mathrm{o}(1)) \qquad \text{as } r = |x| \to \infty, \ \frac{x}{r} = -\theta. \tag{15}$$

From (12) it follows that

$$lw = 0 \quad \text{in } \Omega = R^3 \backslash D, \qquad lw = 0 \quad \text{in } D \tag{16}$$

$$w = o(r^{-1}) \qquad \text{as } r \to \infty. \tag{17}$$

Equation (17) follows from (11) and (15).

The first equation (16) and (17) imply that $w=0$ in $\Omega$ (see [4], p 25). Thus

$$w=0 \quad \text{on } \Gamma. \tag{18}$$

This and the second equation (16) imply that $w=0$ in $D$ since $k^2 \notin \sigma(l_{\mathrm{D}})$. Since $w=0$ in $D$ and in $\Omega$ it follows from the jump relation for the normal derivative $w$ across $\Gamma$ that $\eta=0$. Therefore (10) holds.

A proof of (10) that does not use the assumption $k^2 \notin \sigma(l_0)$ is given in [3].

Let $\psi$ be the solution to (1) with properties (6), (7). Let us take a linear combination of $u(x, \theta, k)$ which approximates $\psi$ in $L^2(D)$:

$$\|\psi - \psi_\varepsilon\|_{L^2(D)} \to 0, \quad \varepsilon \to 0, \qquad \psi_\varepsilon = \int_{S^2} u(x, \theta, k)\, \mathrm{d}\mu_\varepsilon(\theta). \tag{19}$$

Then

$$-4\pi \int_{S^2} A(\theta', \theta, k)\, \mathrm{d}\mu_\varepsilon(\theta) = \int \mathrm{d}x\, q(x) \exp(-\mathrm{i}k\theta' \cdot x)\psi_\varepsilon. \tag{20}$$

Passing to the limit $\varepsilon \to 0$ one obtains that integrals (5) are known. This completes step 1.

*Step 4.* Suppose $q_1$ and $q_2$ have the same data $A_1(\theta', \theta, k) = A_2(\theta', \theta, k) := A(\theta', \theta, k)$ for all $\theta'$, $\theta$ and a fixed $k>0$. Then

$$u_1 = u_2 \quad \text{in } \Omega = R^3 | D \tag{21}$$

Indeed, let $\varphi := u_1 - u_2$. Then

$$(\Delta + k^2)\varphi = 0 \quad \text{in } \Omega, \qquad k^2 > 0 \tag{22}$$

$$\varphi = o(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{23}$$

Equation (23) follows from the assumption $A_1 = A_2$. It follows from (22) and (23) that $\varphi = 0$ in $\Omega$ [4, p 25]. Thus $u_1 = u_2$ in $\Omega$ as claimed.

If one argues as in step 1, one obtains the equation

$$0 = \int \exp(-\mathrm{i}k\theta' \cdot x)(q_1\psi_1 - q_2\psi_2)\, \mathrm{d}x. \tag{24}$$

Here

$$\psi_1 = \lim_{\varepsilon \to 0} \psi_{1\varepsilon}. \tag{25}$$

$\psi_{1\varepsilon}$ is constructed as in formula (19) with $u = u_1$ and $\mu_{1\varepsilon}$ in place of $\mu_\varepsilon$, and $\psi_2$ is constructed as

$$\psi_2 = \lim_{\varepsilon \to 0} \psi_{2\varepsilon} \qquad \psi_{2\varepsilon} := \int_{S^2} u_2(x, \theta, k)\, \mathrm{d}\mu_{1\varepsilon} \tag{26}$$

with the same $\mu_{1\varepsilon}$ as for $\psi_1$.

By the construction $\psi_1$ is of the form (6), (7) with $R=R_1$. We claim that $\psi_2$ defined by (26) is also of the form (6), (7) with $R=R_2$

In order to prove this, note that

$$l_2\psi_2:=(\nabla^2+k^2-q_2(x))\psi_2=0 \quad \text{in } R^3 \tag{27}$$

and

$$\psi_2=\psi_1 \quad \text{in } \Omega. \tag{28}$$

The last equation follows from (21) and the definition of $\psi_1$ and $\psi_2$. It follows from the argument given in [1] that

$$R_1=L^{-1}f, \quad \operatorname{supp} f\subset D \tag{29}$$

where $f$ is a certain function, $f\in L^2(D)$,

$$L^{-1}f:=-\int\frac{\tilde{f}(\lambda)\exp(\mathrm{i}\lambda\cdot x)\,\mathrm{d}\lambda}{\lambda^2+2k\theta\cdot\lambda} \qquad \theta\cdot\theta=1, \quad \theta\in\mathbb{C}^3 \tag{30}$$

and $\tilde{f}(\lambda):=(2\pi)^{-3}\int f(x)\exp(-\mathrm{i}\lambda\cdot x)\,\mathrm{d}x$.

From (28) and (29) it follows that

$$R_2=L^{-1}f \quad \text{in } \Omega, \quad \operatorname{supp} f\subset D. \tag{31}$$

Let $\Phi$ be the (unique) solution to (27) of the form (6), (7) with $R=r$. Then

$$r=L^{-1}h, \quad h\in L^2(D), \quad \operatorname{supp} h\subset D \tag{32}$$

and both $r$ and $R_2$ solve the same equation

$$Lr:=(\nabla^2+2\mathrm{i}k\theta\cdot\nabla)r=q_2r+q_2 \quad \text{in } R^3. \tag{33}$$

We claim that equation (33) has at most one solution which satisfies the condition (31) with some $f\in L^2(D)$. In other words one has

*Lemma A. If*

$$Lr=qr \quad \text{in } R^3 \qquad r\in L^2_{\mathrm{loc}} \tag{34}$$

*and*

$$r=L^{-1}f \quad \text{in } \Omega \tag{35}$$

*where $f$ is an arbitrary function such that*

$$f\in L^2(D), \quad \operatorname{supp} f\subset D \tag{36}$$

*then $r=0$ and $f=0$.*

The proof of the lemma will be given later. It follows from the lemma that

$$r=R_2 \quad \text{in } R^3. \tag{37}$$

This means that $\Phi=\psi_2$ in $R^3$. Thus, $\psi_2$ has property (6), (7) with $R=R_2$. This and (24) imply, as in step 3, that

$$\tilde{q}_1-\tilde{q}_2=0. \tag{38}$$

Thus $q_1 = q_2$. Step 4 is finished provided we prove lemma A.

*Proof of lemma A.* It follows from (34) that

$$r = L^{-1} q r + \alpha \tag{39}$$

where $\alpha$ is an arbitrary solution to

$$L\alpha = 0 \quad \text{in } R^3, \qquad \alpha \in L^2_{\text{loc}}. \tag{40}$$

It follows from (39) and (35) that

$$\alpha = L^{-1} H \quad \text{in } \Omega, \qquad H \in L^2(D), \quad \operatorname{supp} H \subset D. \tag{41}$$

Thus

$$\alpha = L^{-1} H + h_1 \quad \text{in } R^3, \qquad h_1 \in L^2(D), \quad \operatorname{supp} h_1 \subset D. \tag{42}$$

Fourier transform (42) and (40) to get

$$\tilde{\alpha} = L^{-1}(\lambda) \tilde{H} + \tilde{h}_1 \tag{43}$$

$$L(\lambda)\tilde{\alpha} = 0 \qquad L(\lambda) := \lambda^2 + 2ik\theta \cdot \lambda. \tag{44}$$

It follows from (44) that

$$\tilde{\alpha} = 0 \quad \text{if } \lambda \notin M := \{\lambda : L(\lambda) = 0\} \tag{45}$$

Let $\lambda_0 \in M$, $\lambda \notin M$, $\lambda \to \lambda_0$. It follows from (43) that

$$\lim_{\lambda \to \lambda_0} L^{-1}(\lambda) \tilde{H} = -\tilde{h}_1(\lambda_0) \tag{46}$$

where the limit $\lim_{\lambda \to \lambda_0} \tilde{h}_1 = \tilde{h}_1(\lambda_0)$ exists and is finite since $\tilde{h}_1$ is an entire function of $\lambda$. Since $\tilde{H}(\lambda)$ is entire, (46) implies that

$$\tilde{H}(\lambda) = L(\lambda) b(\lambda) \tag{47}$$

where $b(\lambda)$ is also entire. This and (43) imply that $\tilde{\alpha}$ is entire. Since $\tilde{\alpha}$ vanishes outside $M$ it vanishes on an open set in $\mathbb{C}^3$. Thus $\tilde{\alpha} = 0$ and $\alpha = 0$. Lemma A is proved.

This completes the proof of the uniqueness theorem.

*Theorem. If $A(\theta', \theta, k)$ is known for all $\theta', \theta \in S^2$ and a fixed $k > 0$ then the real-valued compactly supported $q \in L^2(D)$ is uniquely determined.*

### *2.3.*

Let us give another proof of the uniqueness theorem, a shorter one but less constructive.

Assume that $q_j \in L^2(D)$, $q_j = 0$ outside $D$, $q_j$ are real-valued, $j = 1, 2$, and $q_j$ produce the same scattering amplitude $A(\theta', \theta, k)$ for all $\theta', \theta \in S^2$ and for a fixed $k > 0$. We wish to prove that this implies $q_1 = q_2$. Let $l_j := \Delta + k^2 - q_j(x)$, $l_j u_j = 0$. One has $l_1 u_1 - l_2 u_2 = l_1 u - q u_2 = 0$, where $u := u_1 - u_2$, $q := q_1 - q_2$. Thus (*) $l_1 u = q u_2$. Take an arbitrary scattering solution, i.e the solution to (1)–(3) with $q = q_2$, $u_2 \in N_D(l_2)$ which is defined uniquely by specifying the unit vector $\theta$ (see (1)–(3)) and let $u_1 \in N_D(l_1)$ be the scattering solution corresponding to the same $\theta$. Then $u = 0$ in $\Omega$ as proved in step 4. Thus (**) $u = u_N = 0$ on $\Gamma = \partial D$. Let $w_1 \in N_D(l_1)$ be an arbitrary scattering solution.

Multiply (*) by $w_1$, integrate over $D$ and then by parts to get (***)$\int_D q u_2 w_1 \, dx = 0$, where (**) was used. The set $\{u_2 w_1\}$ where $u_2(w_1)$ runs through all the scattering solutions for $l_2(l_1)$ is complete in $L^2(D)$ [1, 3]. Therefore (***) implies $q = 0$, i.e. $q_1(x) = q_2(x)$. The proof is complete.

## 3. Numerical recovery of $q(x)$

Let us assume without loss of generality that $k = 1$ and denote $A(\theta', \theta) := A(\theta', \theta, k)$. Given $A(\theta', \theta)\ \forall\theta', \theta \in S^2$ one can write explicitly the scattering solution (1)–(3) in the region $\Omega_a := R^3 \backslash B_a$, $B_a := \{x: |x| \leq a\}$:

$$u := u(x, \theta) = \exp(i\theta \cdot x) + \sum_{n=0}^{\infty} A_n(\theta) Y_n(\theta') h_n(r) \qquad r > a. \tag{48}$$

Here $B_a$ is the smallest ball containing the support of $q(x)$, $Y_n(\theta)$ are the spherical harmonics orthonormalised in $L^2(S^2)$, $r = |x|$, $\theta' = xr^{-1}$, $h_n(r)$ are the spherical Hankel functions normalised so that $h_n(r) \sim r^{-1}\exp(ir)$ as $r \to +\infty$, $A_n(\theta) := (A(\theta, \theta'), Y_n(\theta'))_{L^2(S^2)}$. Formula (48) is obtained in [5]. It follows from the fact that the solution to (1)–(3) and the function (48) both solve equation (22) in $\Omega_a$ and their difference satisfies (23), so that they are identical in $\Omega_a$. The series (48) can be termwise differentiated in $\Omega_a$. Thus it gives the knowledge of the set $\{u, u_N\}\forall\theta \in S^2$ on $S_R := \{x: |x| = R\}$, where $R > a$ is an arbitrary number, $u_N = u_r$ is the normal derivative of $u$ on $S_R$. Since the set $\{u\}\forall\theta \in S^2$ of the scattering solutions is complete in $N_{D_1}(l)$ in $L^2(D_1)$ (and in $H^1(D_1)$) where $D_1 \subset R^3$ is an arbitrary compact domain [3], the knowledge of the set $\{u, u_N\}\forall\theta \in S^2$ implies the knowledge of the set $\{w, w_N\}|_{S_R}\forall w \in N_{B_R}(l)$. This knowledge allows one to recover $q(x)$ uniquely [6, 3]. Thus we have given the third proof of the theorem. Our goal, however, is to give a constructive method for finding $q(x)$ given $A(\theta', \theta)$.

To achieve this goal let us note that from (20), (6) and (7) it follows that

$$\tilde{q}(p) = -4\pi \lim_{\substack{\theta - \theta' = p \\ |\theta| \to \infty,\ \theta \cdot \theta = 1}} \left( \lim_{\epsilon \to 0} \int A(\theta', \alpha)\, d\mu_\epsilon(\alpha) \right). \tag{49}$$

Here $p \in R^3$ is an arbitrary vector, $\tilde{q}(p) := \int q(x) \exp(ip \cdot x)\, dx$, $\mu_\epsilon(\alpha) = \mu_\epsilon(\alpha, \theta)$ is the signed measure with the property

$$\left\| \psi(x, \theta) - \int_{S^2} u(x, \alpha)\, d\mu_\epsilon(\alpha) \right\|_{L^2(D_1)} \to 0 \qquad \text{as} \quad \varepsilon \to 0 \tag{50}$$

where $D_1 \subset R^3$ is an arbitrary compact domain, and $\psi(x, \theta)$ is defined in (6), (7). The existence of such a solution to equation (1) is proved in [1] for $q \in L^\infty(D)$ and in [2] for $q \in L^2(D)$. In (49) one can use the $d\mu_\epsilon(\alpha)$ which is of the form $d\mu_\epsilon = h_\epsilon(\alpha)\, d\alpha$, where $h_\epsilon(\alpha) \in L^2(S^2)$, $d\alpha$ is the element of the area of $S^2$. Another possibility is to use $d\mu_\epsilon = \Sigma_{j=1}^{n(\epsilon)} c_j \delta(\alpha - \alpha_j)\, d\alpha$, where $\delta(\alpha - \alpha_j)$ are the delta functions, $c_j = \text{constant}$, $\alpha_j \in S^2$, and the set $\{\alpha_j\}$ is dense in $S^2$.

In order to use formula (49) for computations one needs a method for computing $d\mu_\epsilon$ with the property (50). Let us give such a method.

In what follows we take $d\mu_\varepsilon = h_\varepsilon(\alpha)\, d\alpha$, $h_\varepsilon(\alpha) \in L^2(S^2)$. Consider the variation problem

$$J_\varepsilon = \min, \quad J_\varepsilon = J_\varepsilon(h) := \left\| \exp(-i\theta\cdot x)\int_{S^2} u(x,\alpha)h(\alpha)\,d\alpha - 1 \right\|_\varepsilon . \tag{51}$$

Here $\varepsilon^{-1} > a$

$$\|f\|_\varepsilon^2 := \int_{a<|x|<\varepsilon^{-1}} |f|^2(1+|x|)^{-\beta}\,dx \qquad 1 \leqslant \beta < 2. \tag{52}$$

Since $\inf_h J_\varepsilon := m_\varepsilon \geqslant 0$, there exists a minimising sequence $h_{\varepsilon,\delta}$, $J_\varepsilon(h_{\varepsilon,\delta}) \to m_\varepsilon$ as $\delta \to 0$. One knows that there is a sequence $H_n(\alpha) \in L^2(S^2)$ such that

$$\left\| \psi(x,\theta) - \int_{S^2} H_n(\alpha)u(x,\alpha)\,d\alpha \right\|_{L^2(D_1)} \to 0 \quad \textit{as } n \to \infty. \tag{53}$$

Since $\psi = \exp(i\theta\cdot x)(1 + R(x,\theta))$, one denotes

$$\int_{S^2} h(\alpha)u(x,\alpha)\,d\alpha := \psi_h \qquad r_h := \exp(-i\theta\cdot x)\psi_h - 1$$

and writes

$$\begin{aligned}\|\psi - \psi_{H_n}\|_{L^2(D_1)} &= \|\exp(i\theta\cdot x)[1 + R - \exp(-i\theta\cdot x)\psi_{H_n}]\|_{L_2(D_1)} \\ &= \|\exp(i\theta\cdot x)(R - r_{H_n})\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus

$$\|R - r_n\|_{L^2(D_1)} \to 0 \quad \text{as } n \to \infty, \qquad r_n := r_{H_n}.$$

One knows that

$$\|R\| := \left( \int_{R^3} |R|^2 (1+|x|)^{-\beta}\,dx \right)^{1/2} < \frac{c}{|\theta|} \qquad 1 \leqslant \beta < 2 \quad \theta \geqslant 1 \tag{54}$$

where $c > 0$ is an absolute constant, $\theta\cdot\theta = 1$, $\theta \in \mathbb{C}^3$. Therefore for any fixed $\varepsilon > 0$ and $\theta \in \mathbb{C}^3$, $\theta\cdot\theta = 1$, one can find $r_n = r_n(x,\varepsilon,\theta)$ such that

$$\|r_n\|_\varepsilon \leqslant \frac{c}{|\theta|} \qquad |\theta| \geqslant 1, \quad \theta\cdot\theta = 1. \tag{55}$$

Note that $J_\varepsilon = \|r_h\|_\varepsilon$. Let $r_\delta := r_{h_{\varepsilon,\delta}}$, so that $J(h_{\varepsilon,\delta}) \to m_\varepsilon$ as $\delta \to 0$. Since $h_{\varepsilon,\delta}$ is a minimising sequence, one has $J(h_{\varepsilon,\delta}) \leqslant J(H) + \eta$, for a fixed $H \in L^2(S^2)$ and an arbitrary small $\eta$ if $\delta > 0$ is small enough. Therefore, if $H_n$ corresponds to $r_n$ which satisfies (55), there exists a $\delta = \delta(\varepsilon,\theta)$ such that $h_\varepsilon := h_{\varepsilon,\delta(\varepsilon,\theta)}$ satisfies the estimate

$$J(h_\varepsilon) \leqslant \frac{c+1}{|\theta|} \qquad \theta\cdot\theta = 1, \quad |\theta| \geqslant 1 \tag{56}$$

where $c$ is an absolute constant that does not depend on $\varepsilon$ or $\theta$. Note that $h_\varepsilon(\alpha)$ depends on $\theta$.

*Lemma B. The constructed sequence* $h_\varepsilon(\alpha)$ *satisfies (50) with* $\mathrm{d}\mu_\varepsilon = h_\varepsilon(\alpha)\,\mathrm{d}\alpha$.

*Proof.* Equation (50) is equivalent to

$$\|R - r_\varepsilon\|_{L^2(D_1)} \to 0 \quad \text{as } \varepsilon \to 0 \tag{57}$$

where $r_\varepsilon := \exp(-\mathrm{i}\theta \cdot x)\int_{S^2} u(x, \alpha)h_\varepsilon(\alpha)\,\mathrm{d}\alpha - 1$. Therefore it is sufficient to prove (57). Inequality (56) can be written as

$$\|r_\varepsilon\|_\varepsilon \leqslant \frac{c+1}{|\theta|} \qquad \theta \cdot \theta = 1, \quad |\theta| \geqslant 1. \tag{58}$$

Select a subsequence $r_{\varepsilon_n}$ that is weakly convergent in $L^2_{\mathrm{loc}}(\Omega_a)$ as $\varepsilon \to 0$, which we denote again $r_\varepsilon$. Note that equation (1) with $k=1$ implies that

$$Lr_\varepsilon := (\Delta + 2\mathrm{i}\theta \cdot \nabla)r_\varepsilon = qr_\varepsilon + q \quad \text{in } R^3. \tag{59}$$

We will apply to $r_\varepsilon$ the known elliptic estimate

$$\|r_\varepsilon\|_{H^2(D_1)} \leqslant c(D_1, D_2)(\|Lr_\varepsilon\|_{L^2(D_2)} + \|r_\varepsilon\|_{L^2(D_2)}) \tag{60}$$

where $D_1$ is a strictly inner subdomain of $D_2$, $D_2 \subset R^3$ is an arbitrary compact domain, and the constant $c(D_1, D_2)$ depends on $D_1$, $D_2$ and $\theta$ but does not depend on $r_\varepsilon$. If $q \in L^2(D)$ then the solution to the elliptic equation (59) is in $H^2_{\mathrm{loc}}$ so that $r_\varepsilon \in L^\infty_{\mathrm{loc}}$. Thus, the right-hand side of (60) is bounded since $\|r_\varepsilon\|_{L^2(D_1)} \leqslant c(D_2)$ because of (58). Thus $r_\varepsilon \in H^2_{\mathrm{loc}}(R^3)$. By the embedding theorem this implies that $r_\varepsilon$ converges in $H^m_{\mathrm{loc}}(R^3)$, $m<2$. This, equation (59) and estimate (60) imply that $r_\varepsilon$ converges in $H^2_{\mathrm{loc}}(R^3)$. Let $r_0 := \lim_{\varepsilon\to 0} r_\varepsilon$ in $H^2_{\mathrm{loc}}$. Pass to the limit $\varepsilon \to 0$ in (59) to get

$$Lr_0 = qr_0 + q \quad \text{in } R^3. \tag{61}$$

Pass to the limit $\varepsilon \to 0$ in (58) (using Fatou's lemma) to get

$$\|r_0\| \leqslant \frac{c+1}{|\theta|} \qquad \theta \cdot \theta = 1, \quad |\theta| \geqslant 1 \tag{62}$$

where the norm is defined in (54). To complete the proof it is sufficient to show that $r_0 = R$. Indeed, this would give (57) (which is equivalent to (50)) for the chosen subsequence, but since the limit of any subsequence is the same, namely $R$, this also gives convergence of the whole sequence $r_\varepsilon$ in $H^2_{\mathrm{loc}}$ and in the norm (52).

In order to prove that $r_0 = R$ note that $R$ satisfies equation (61). Define $\rho := R - r_0$. Then

$$L\rho = q\rho \quad \text{in } R^3, \quad \|\rho\| < \infty. \tag{63}$$

We claim that (63) implies $\rho = 0$. This is proved in lemma C below. The proof of lemma B is complete.

*Lemma C. If* $\rho$ *satisfies* (63) *then* $\rho = 0$.

*Proof.* Write (63) as

$$\rho = L^{-1}(q\rho) + w. \tag{64}$$

Here $w$ is an arbitrary solution to $Lw = 0$ and $L^{-1}$ is defined by (30). The rest of the argument is close to the one in the proof of lemma A in §2. Instead of the condition

$\operatorname{supp} \rho \subset D$ we now have the condition $\|\rho\| < \infty$. The conclusion of lemma C follows from the results in [3] but we give a proof for convenience of the reader. It is sufficient to prove that $w = 0$. Indeed, if $w = 0$ then (64) implies

$$\|\rho\|_{L^\infty(D)} = \|L^{-1}(q\rho)\|_{L^\infty(D)} \leqslant c\varepsilon(\theta)\|q\rho\|_{L^2(D)} \leqslant c\varepsilon(\theta)\|q\|_{L^2(D)}\|\rho\|_{L^\infty(D)} \tag{65}$$

where the first inequality is proved in [2] with $\varepsilon(\theta) \to 0$ as $|\theta| \to \infty$, $\theta \cdot \theta = 1$, $\theta \in \mathbb{C}^3$. Therefore if $|\theta| \gg 1$, $\theta \cdot \theta = 1$, (65) implies that $\rho = 0$.

Let us prove that $w = 0$. This will conclude the proof of lemma C. Take the Fourier transform of the equation $Lw = 0$ in $R^3$ to get

$$L(\lambda)\tilde{w} = 0 \qquad L(\lambda) := \lambda^2 + 2\lambda \cdot \theta, \quad \theta \cdot \theta = 1. \tag{66}$$

The set $L(\lambda) = 0$, $\lambda \in R^3$ is a circle $C$ (see [1] for a detailed discussion). Since $\|\rho\| < \infty$ by assumption and $\|L^{-1}(q\rho)\| < \infty$ for $q\rho \in L^2(D)$, $q\rho = 0$ outside $D$ (see [8]), one has

$$\limsup_{R\to\infty} \int_{|x|\leqslant R} |w|^2(1+|x|)^{-\beta}\, dx < c.$$

It follows ([7], theorem 7.1.27) that

$$\int_C |\tilde{w}|^2\, ds \leqslant c \limsup_{R\to\infty} R^{-2} \int_{|x|\leqslant R} |w|^2\, dx \leqslant c \limsup_{R\to\infty} \int_{|x|\leqslant R} \frac{|w|^2}{(1+|x|)^\beta} \frac{1+|x|^\beta}{R^2}\, dx = 0$$

if $\beta < 2$. Thus $\tilde{w} = 0$ and $w = 0$. Lemma C is proved.

We give a summary of the numerical method for the recovery of $q(x)$ given $A(\theta', \theta)$ for all $\theta'$, $\theta$, $k = 1$.

(1) Fix arbitrary $p \in R^3$ and $\theta \in \mathbb{C}^3$, $\theta \cdot \theta = 1$, $|\theta| \gg 1$. Find $\theta' \in \mathbb{C}^3$, $\theta' \cdot \theta' = 1$ so that $\theta - \theta' = p$.

(2) Take a sequence $\varepsilon \to 0$. For each $\varepsilon$ find an $h_\varepsilon(\alpha)$ such that $J_\varepsilon(h_\varepsilon) \leqslant c$, where $c$ is a constant that does not depend on $\varepsilon$ as $\varepsilon \to 0$.

(3) Compute an approximate value of $\tilde{q}(p)$ by the formula (cf (49))

$$\tilde{q}(p) \approx -4\pi \int_{S^2} A(\theta', \alpha) h_\varepsilon(\alpha)\, d\alpha. \tag{67}$$

In (67) one uses the value of $A(\theta', \alpha)$ for complex $\theta'$, $\theta' \cdot \theta' = 1$. The given values of $A(\theta', \alpha)$, $\theta'$, $\alpha \in S^2$, determine $A(\theta', \alpha)$ uniquely for all $\theta' \in \mathbb{C}^3$, $\theta' \cdot \theta' = 1$ (see [4], p 62]). Practically one can use the truncated Fourier series for $A(\theta', \alpha)$ in spherical harmonics:

$$A(\theta', \alpha) = \sum_{n=0}^{N} A_n(\alpha) Y_n(\theta') \tag{68}$$

$$A_n(\alpha) := (A(\theta', \alpha), Y_n(\theta'))_{L^2(S^2)}. \tag{69}$$

If $q(x)$ has compact support then $A(\theta', \alpha)$ is entire function of $\theta'$, and therefore the full (not truncated) series (68) converges for all $\theta' \in \mathbb{C}^3$. This means that the coefficients $A_n(\alpha)$ decay very rapidly as $n$ grows. For the spherical harmonics the following estimate is known [9, p 261]:

$$|D^r Y_n(\theta)| = O(n^{3/2-1+|r|}) \qquad n \to \infty. \tag{70}$$

Here $D^r$ is a derivative of order $r$, $r$ is a multi-index, and the estimate is written for spherical harmonics in $R^3$. In $R^d$ the right-hand side of (70) is O $(n^{d/2-1+|r|})$.

The values of $A(\theta', \alpha)$ for complex $\theta'$, $\theta' \cdot \theta' = 1$, can be computed by formula (68) since the functions $Y_n(\theta')$ are well defined for such complex $\theta'$.

The suggested numerical method is not easy to implement. But there cannot be a numerical method for solving the inverse problem (IP) of finding $q(x)$ given $A(\theta', \theta)$ that would be easy to implement. The reason is that this inverse problem is highly ill-posed since any small perturbation of $A(\theta', \theta)$ may lead to a function that is not a scattering amplitude. Since there were no numerical methods for solving IP which would be theoretically justified it is interesting to do numerical experiments with the method suggested and theoretically justified in this paper.

A necessary and sufficient condition for a given function $A(\theta', \theta)$, $\theta'$, $\theta \in S^2$ to be the scattering amplitude corresponding to a compactly supported real-valued $q(x) \in L^2(D)$ is given in [3].

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