Inverse Problems **4** (1988) L1–L5. Printed in the UK

LETTER TO THE EDITOR

A uniqueness theorem for a boundary inverse problem

A G Ramm

Mathematics Department, Kansas State University, Manhattan, KS 66506, USA

Received 29 June 1987

Abstract. Let $D \subset R^3$ be a bounded domain with a smooth boundary Γ,

$$-\Delta u + q(x)u = 0 \text{ in } D \qquad u = f, \quad u_N = h \text{ on } \Gamma$$

and $q(x) \in L^\infty(D)$. From knowledge of the set $\{f, h\}$ where f runs through $C^1(\Gamma)$ the coefficient $q(x)$ is uniquely recovered. Analytical formulae for $q(x)$ are given. Applications are considered.

Let $D \subset R^3$ be a bounded domain with a smooth boundary Γ, $q(x) \in L^\infty(D)$,

$$-\Delta u + q(x)u = 0 \text{ in } D \qquad u = f, \quad u_N = h \text{ on } \Gamma \tag{1}$$

where Δ is the Laplacian and N is the outward normal to Γ. Assume that zero is not an eigenvalue of the Dirichlet operator $-\Delta + q$ in D. If q and f are known then u is uniquely defined in D. Therefore h is uniquely defined. We are interested in the inverse problem A: given the set of pairs $\{f, h\}$ for f running through $C^1(\Gamma)$, determine $q(x)$.

Problems of this type were studied recently (see [1, 2] and references therein). In [1] the uniqueness of the solution to the inverse problem B of finding $\sigma(x) \in C^\infty(\bar{D})$, $\sigma(x) > 0$ in $\bar{D}$, $\bar{D}$ is the closure of D, from the set $\{f_0, \sigma h_0\}$, where $\nabla \cdot (\sigma(x)\nabla w) = 0$ in D, $w = f_0$, $w_N = h_0$ on Γ, is proved. Some additional results are obtained in [2].

Our purpose is to show that some uniqueness theorems for inverse problems can be easily obtained by the method given in [3, 4]. As an example we give a short (but nevertheless complete) proof of the following.

Theorem 1. The set $\{f, h\}$ where f runs through $C^1(\Gamma)$ determines $q(x) \in L^\infty(D)$ uniquely.

Remark 1. It is known that problems A and B are closely related: put $w = \sigma^{-1/2}(x)u$ then u solves (1) with $q(x) = \sigma^{-1/2}\Delta\sigma^{1/2}$. If σ and σ_N are known on Γ then knowledge of the set $\{f_0, \sigma h_0\}$ implies knowledge of the set $\{f, h\}$. In applications problem B is the problem of finding the conductivity of a body from measurements of the potential and current on its surface.

Remark 2. Some uniqueness theorems in inverse problems of geophysics are given in [5]. The ideas in [1] are applied to some inverse scattering problems in [6].

Proof of theorem 1. We will use the result which is a particular case of proposition 1 from [4]: there exists a solution to equation (1) of the form

$$u(x, z) = \exp(\mathrm{i}z \cdot x)(1 + R(x, z)) \tag{2}$$

where $z \in \mathbb{C}^3$, $z \cdot z := z_1^2 + z_2^2 + z_3^2 = 0$, and

$$\|R(x, z)\|_{L^2(D)} \leqslant c|z|^{-1/2} \qquad \text{as } |z| \to \infty,\ z \cdot z = 0. \tag{3}$$

Here $|z| = (|z_1|^2 + |z_2|^2 + |z_3|^2)^{1/2}$ and $c = \text{constant} > 0$; c does not depend on z but depends on $\|q\|_{L^\infty(D)}$. A similar result was first proved in [1].

Let $\lambda \in \mathbb{C}^3$, $\lambda \cdot \lambda = 0$. We have

$$\int_D \exp(\mathrm{i}\lambda \cdot x) qu \, \mathrm{d}x = \int_D \exp(\mathrm{i}\lambda \cdot x) \Delta u \, \mathrm{d}x$$

$$= \int_\Gamma \left(\exp(\mathrm{i}\lambda \cdot x) h - f \frac{\partial}{\partial N} \exp(\mathrm{i}\lambda \cdot x) \right) \mathrm{d}s := F(z, f, h) \tag{4}$$

where F is known by assumption. Substitute (2) into (4) and let $|z| \to \infty$ while keeping the following conditions satisfied:

$$\lambda \cdot \lambda = 0 \qquad z \cdot z = 0 \qquad \lambda + z = p \tag{5}$$

where $p \in R^3$ is an arbitrary given vector. Note that (5) are ten equations for twelve parameters: λ and z are each determined by six real numbers. It is not difficult to check that conditions (5) can be satisfied (for any given $p \in R^3$ and $|z| \to \infty$). This is done in detail in [4]. If $|z| \to \infty$, by (3), the left-hand side of (4) becomes $\int q(x) \exp(\mathrm{i}p \cdot x) \, \mathrm{d}x := \tilde{q}(p)$. Since the right-hand side F of (4) is known we found $\tilde{q}(p)$ from the data. Namely, by the assumption the pairs $\{f, h\}$ corresponding to the solutions (2) with any $z \in \mathbb{C}^3$, $z \cdot z = 0$, are known, so that F in (4) is known. If $\tilde{q}(p)$ is found, $q(x)$ is obtained uniquely and effectively by taking the inverse Fourier transform. Theorem 1 is proved.

Corollary 1. If σ and σ_N on Γ are known, $0 < c \leqslant \sigma(x)$, $x \in \bar{D}$ and $\sigma(x) \in W^{2,\infty}(D)$, then problem B has at most one solution. Furthermore, this solution can be effectively constructed as follows:

(i) solve problem A with $q(x) = \sigma^{-1/2} \Delta \sigma^{1/2}$ (see remark 1);

(ii) find $\sigma(x)$ by solving the problem

$$\Delta \sigma^{1/2} - q(x) \sigma^{1/2} = 0 \text{ in } D \tag{6}$$

$$\sigma^{1/2} \text{ and } (\sigma^{1/2})_N \text{ are known on } \Gamma. \tag{7}$$

This is a Cauchy problem for the elliptic equation (6). By the uniqueness of the solution of the Cauchy problem for this equation one concludes that $\sigma^{1/2}$ is uniquely determined. For the compatible Cauchy data (7) $\sigma^{1/2}$ can actually be found as the solution to the Dirichlet problem

$$\Delta \sigma^{1/2} - q(x) \sigma^{1/2} = 0 \text{ in } D \qquad \sigma^{1/2} \text{ is known on } \Gamma. \tag{8}$$

Remark 3. Problem B has been discussed in [8] and it was proved that the set $\{f_0, \sigma h_0\}$ determines σ and all its derivatives on Γ provided that $\sigma \in C^\infty(\bar{D})$ and $\Gamma \in C^\infty$.

Remark 4. One of the results in [6] is the following 'n-dimensional Borg–Levinson theorem': let $q_j(x) \in C^\infty(\bar{D})$, $\operatorname{Im} q_j = 0$, $j = 1, 2$, and suppose that $\lambda_m^{(1)} = \lambda_m^{(2)}$, and $\psi_{mN}^{(1)} = \psi_{mN}^{(2)}$ on Γ for all $m = 1, 2, \ldots$. Then $q_1(x) = q_2(x)$ in D. Here

$$(-\Delta + q_j(x) - \lambda_m^{(j)})\psi_m^{(j)} = 0 \text{ in } D \qquad \psi_m^{(j)} = 0 \text{ on } \Gamma$$

$$\|\psi_m^{(j)}\|_{L^2(D)} = 1 \qquad \psi_{mN} := \frac{\partial \psi_m}{\partial N} \text{ on } \Gamma.$$

Some general results of this type are announced in [7]. One can use theorem 1 to prove such a result.

For example, if $-\Delta G + q(x)G = \delta(x-y)$ in D, $G = 0$ on Γ, then

$$G = \sum_{m=1}^{\infty} \lambda_m^{-1} \psi_m(x) \overline{\psi_m(y)},$$

where the bar stands for complex conjugate. If

$$-\Delta u + q(x)u = 0 \text{ in } D \qquad u = f \text{ on } \Gamma \tag{9}$$

then

$$u = -\sum_{m=1}^{\infty} \lambda_m^{-1} \psi_m(x) f_m \qquad f_m := \int_\Gamma f \, \overline{\frac{\partial \psi_m}{\partial N}} \, ds \tag{10}$$

and

$$h := u_N = -\sum_{m=1}^{\infty} \lambda_m^{-1} f_m \frac{\partial \psi_m}{\partial N}. \tag{11}$$

Therefore, if the data $\{\lambda_m, \partial\psi_m/\partial N|_\Gamma\}_{m=1}^\infty$ are given then the set $\{f, h\}$ is given. This set determines $q(x)$ uniquely by theorem 1. The argument in this remark is not quite complete yet since we did not check that the series (10) can be termwise differentiated. A similar idea was used with the Green function G_λ, $(-\Delta + q - \lambda)G_\lambda = \delta(x-y)$, in place of G and a more complicated justification of the possibility to termwise differentiate certain series is given in [6].

One can give a justification of the argument as follows. If $f_m = 0$ for all m sufficiently large then the series (10) can be termwise differentiated since it is a finite sum. The set of f with f_m vanishing for all sufficiently large m is dense in $L^2(\Gamma)$ since the set $\{\partial\psi_m/\partial N\}_{m=1}^\infty$ is complete in $L^2(\Gamma)$ [5]. The mapping Λ: $f \rightarrow h$ is known to be continuous from $H^n(\Gamma)$ into $H^{n-1}(\Gamma)$ for any real n if $q \in C^\infty(\bar{D})$ and $\Gamma \in C^\infty$, where $H^n(\Gamma)$ is the Sobolev space. Therefore if the set $\{f, h\}$ is known for f running through a dense set in $H^n(\Gamma)$ it is known for any $f \in H^n(\Gamma)$ by passing to the limit. Since $L^2(\Gamma) = H^0(\Gamma)$ and $H^n(\Gamma) \subset H^p(\Gamma)$ for $n > p$, one knows the set $\{f, h\}$ for any $f \in H^n(\Gamma)$, $n > 0$, if one knows this set for any $f \in L^2(\Gamma)$. Since $C^1(\Gamma) \subset L^2(\Gamma)$ one knows the set $\{f, h\}$ for all $f \in C^1(\Gamma)$. This argument shows that the assumptions $q \in C^\infty(\bar{D})$ and $\Gamma \in C^\infty$ can be weakened: a finite smoothness of q and Γ is sufficient.

The problem of convergence of the series in eigenfunctions for elliptic operators of second order is closely connected with the asymptotic behaviour of the spectral function of these operators for large values of spectral parameter. These questions were studied in the literature by B Levitan, L Hörmander and others.

Let us prove a lemma which is used in remark 4.

Lemma. Let $u(x)$ run through the set $M := \{u: u \in C^2(D), u = 0 \text{ on } \Gamma\}$, where $D \subset R^3$ is a bounded domain and $\Gamma = \partial D$ its smooth boundary. Then the set $\{u_N\}$, where $u_N := \partial u/\partial N|_\Gamma$, is dense in $L^2(\Gamma)$. If $M_1 := \{u: u \in C^2(D), u_N = 0 \text{ on } \Gamma\}$ then the set $\{u|_\Gamma\}$ is dense in $L^2(\Gamma)$.

Proof. If the conclusion is wrong then there is an $h \in L^2(\Gamma)$, $h \not\equiv 0$, such that

$$\int_\Gamma \frac{\partial u}{\partial N} h \, ds = 0 \qquad \forall u \in M. \tag{12}$$

Let u solve the problem

$$\Delta u = f \text{ in } D \qquad u|_\Gamma = 0 \tag{13}$$

and $f \in C^a(D)$ is arbitrary, $a = \text{constant} > 0$. By Schauder's estimate $u \in M$. Let v solve the problem

$$\Delta v = 0 \text{ in } D \qquad v|_\Gamma = h. \tag{14}$$

Apply Green's formula:

$$-\int_D vf \, dx = \int_D (u\Delta v - v\Delta u) \, dx = \int_\Gamma (uv_N - vu_N) \, ds = 0 \tag{15}$$

where (12), (13) and (14) are used. Thus, v is orthogonal to any $f \in C^a(D)$. Therefore $v = 0$ and $h = 0$. The last statement of the lemma is proved similarly.

Corollary. Let $\{\psi_m\}$ be the complete orthonormal set of eigenfunctions of the Dirichlet operator $L = -\Delta + q$ in D, $L\psi_m = \lambda_m \psi_m$, $\psi_m|_\Gamma = 0$. Then the linear span of the functions $\{\partial\psi_m/\partial N|_\Gamma\}$ is dense in $L^2(\Gamma)$.

Proof. If $u \in M$ then

$$u = \sum_{m=1}^{\infty} u_m \psi_m \qquad u_m = (u, \psi_m)_{L^2(D)}.$$

The set $\Sigma_{m=1}^n u_m \psi_{mN}$, $n = 1, 2, 3, \ldots$, is dense in the set $\{u_N\}$ and, by the lemma, it is dense in $L^2(\Gamma)$.

Remark 5. Theorem 1 holds for $q(x) \in L^2(D)$ [9].

The author thanks ONR for support, Dr G Uhlmann for sending preprints [2, 6] which influenced the author's work, and Mrs Stanley for typing.

References

[1] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem *Ann. Math.* **125** 153–69

[2] Sylvester J and Uhlmann G 1987 Inverse boundary problems at the boundary—continuous dependence *Preprint*
[3] Ramm A G 1986 Completeness of the products of harmonic functions *Proc. Am. Math. Soc.* **98** 253–6
[4] Ramm A G 1987 Completeness of the products of solutions to PDE and uniqueness theorems in inverse scattering *Inverse Problems* **3** L77–82
[5] Ramm A G 1986 *Scattering by Obstacles* (Dordrecht: Reidel)
[6] Nachman A, Sylvester J and Uhlmann G 1987 An n-dimensional Borg–Levinson theorem *Preprint*
[7] Buhgeim A 1985 Multidimensional inverse problems of spectral analysis *Sov. Math.-Dokl.* **32** 353–5
[8] Kohn R and Vogelius M 1984 Determining conductivity by boundary measurements *Comm. Pure Appl. Math.* **37** 289–97; **38** 643–67
[9] Ramm A G 1987 Uniqueness theorems for multidimensional inverse problems with unbounded coefficients *Preprint*