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LETTER TO THE EDITOR

A uniqueness theorem for two-parameter inversion

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Abstract. A uniqueness theorem for two-parameter inversion is proved within the framework of exact theory.

Let

$$\nabla^2 u + k^2 u + k^2 a_1(x)u + \nabla \cdot (a_2(x)\nabla u) = -\delta(x-y) \qquad \text{in } \mathrm{R}^3. \tag{1}$$

Assume that:

(i) $u(x, y, k)$ is known for all $x, y \in P := \{x{:}x_3 = 0\}$ and $k = k_j$, $j = 1, 2$, $k_j > 0$, $k_1 \neq k_2$;

(ii) $a_1 \in L^\infty(D)$, $a_2 \in W^{2,\infty}(D)$, $1 + a_2(x) > 0$, $a_j(x) = 0$ outside D, $j = 1, 2$, $D \subset R^3_- = \{x{:}x_3 < 0\}$, D is a bounded domain.

Theorem. Under these assumptions $a_j(x)$, $j = 1, 2$, are uniquely determined.

The result in this theorem was proved in [1] in the Born approximation. Here a proof is given within the framework of exact theory. The Born approximation is not used. The method used is developed in [2–6].

Write equation (1) as

$$\nabla \cdot (b_2(x)\nabla u) + k^2 b_1(x)u = -\delta(x-y) \qquad b_j = 1 + a_j, \quad j = 1, 2. \tag{2}$$

Let $y \in P$ and $u = b_2^{-1/2} w$. Then (2) becomes

$$\Delta w + k^2 w - q(x)w + k^2 p(x)w = -\delta(x-y) \tag{3}$$

where Δ is the Laplacian, and

$$q(x) := b_2^{-1/2}(x)\Delta b_2^{1/2}(x) \qquad p(x) := b_1(x)b_2^{-1}(x) - 1. \tag{4}$$

We have used the equation $\delta(x-y)b_2^{-1/2}(x) = \delta(x-y)$ which holds if $y \in P$ since $a_2(y) = 0$. Note that $q(x)$ and $p(x)$ vanish outside D.

The idea of the proof is as follows.

Step 1. The values w on P at $k = k_j$ determine $-q(x) + k_j^2 p(x)$ uniquely.

Step 2. The values of $-q(x) + k_j^2 p(x)$, $j = 1, 2$, $k_1 \neq k_2$, determine $q(x)$ and $p(x)$ uniquely.

Step 3. Given $q(x)$, find $b_2(x)$ and then $b_1(x)$.

Note that $u=w$ on P so that w on P is known. Step 2 is trivial. Step 3 amounts to solving the equation

$$\Delta v - q(x)v = 0 \qquad \text{in } R^3 \tag{5}$$

$$v=1 \text{ outside } D \qquad v:=b^{1/2}(x) \tag{6}$$

or, since $v:=1+d(x)$, $d(x)=0$ outside D,

$$\Delta d - q(x)d = q(x) \quad \text{in } R^3 \qquad d=0 \text{ outside } D. \tag{7}$$

Problem (7) has a unique solution, since the corresponding homogeneous problem has only the trivial solution because of the uniqueness of the solution to the Cauchy problem for elliptic equations (or due to the unique continuation property for elliptic equations: if $\Delta d - q(x)d=0$ and $d=0$ on an open set then $d\equiv 0$; the assumptions on a_j imply $q \in L^\infty(D)$ and this is sufficient for the validity of the unique continuation property).

The basic step 1 is based on the following lemma.

Lemma 1. Let $a(x) \in L^\infty(D)$, $D \subset R^3_-$ is a bounded domain, $k \geqslant 0$ is fixed,

$$\Delta w + k^2 w - a(x)w = -\delta(x-y) \text{ in } R^3 \qquad y \in P = \{x{:}x_3=0\}. \tag{8}$$

The values of $w(x, y)$ for all $x, y \in P$ determine $a(x)$ uniquely.

Proof. Let us write (8) as

$$w(x, y) = g(x, y) + \int g(x, z)a(z)w(z, y)\mathrm{d}z \qquad \int = \int_D \tag{9}$$

where

$$g := (4\pi|x-y|)^{-1} \exp(ik|x-y|). \tag{10}$$

Let $x, y \in P$ in (9). The function $f(x, y) := w - g$ is known on P, so (9) can be written as

$$\int g(x, z)a(z)w(z, y)\mathrm{d}z = f(x, y) \qquad x, y \in P. \tag{11}$$

Multiply (11) by an arbitrary $\varphi(x) \in C_0^\infty(P)$ and integrate over P. The function

$$\int \varphi(x)g(x, z)\mathrm{d}x := m(z) \tag{12}$$

runs through a dense set (in $L^2(D)$) in the set $N_D(\Delta + k^2) := \{m := (\Delta + k^2)m = 0 \text{ in } D\}$ when φ runs through $C_0^\infty(P)$. This statement is proved below as lemma 2. Assuming this, we see that (11) implies

$$\int m(z)a(z)w(z, y)\mathrm{d}z = \int_P f(x, y)\varphi(x)\mathrm{d}x. \tag{13}$$

Similarly, (11) implies

$$\int m(z)a(z)n(z)\mathrm{d}z = a(m, n) := \int_P \int_P f(x, y)\varphi(x)\psi(y)\mathrm{d}x\mathrm{d}y \tag{14}$$

where φ, $\psi \in C_0^\infty(P)$, $m \in N_D(\Delta + k^2)$, $n(z) \in N_D(\Delta + k^2 - a(x)) = \{n : \Delta n + k^2 n - an = 0$ in $D\}$. Therefore the set of integrals is known:

$$\int a(z)m(z)n(z)\mathrm{d}z := a(m, n) \qquad m \in N_D(\Delta + k^2), \quad n \in N_D(\Delta + k^2 - a(x)). \tag{15}$$

This and lemma 3 below imply that the Fourier transform of $a(z)$ is uniquely determined and therefore $a(z)$ is uniquely determined. Lemma 1 is proved. Let us now prove lemmas 2 and 3.

Lemma 2. The set of functions (12) is dense in $L^2(D)$ in $N_D(\Delta + k^2)$.

Without loss of generality assume that k^2 is not an eigenvalue of the Dirichlet Laplacian in D. If it is, then take $D_1 \supset D$, $D_1 \subset R^3_-$ such that k^2 is not an eigenvalue of the Dirichlet Laplacian in D_1 and apply the argument below.

Proof. It is sufficient to prove that the set of $m|_{\Gamma = \partial D}$ is dense in $L^2(\Gamma)$. Suppose it is not. Then there is an $h \in L^2(\Gamma)$, $h \not\equiv 0$, such that

$$\int_\Gamma \mathrm{d}s\, h \int_P \varphi(x) g(x, s)\mathrm{d}x = 0 \qquad \forall \varphi \in C_0^\infty(P). \tag{16}$$

This implies that

$$u(x) := \int_\Gamma \mathrm{d}s\, h(s) g(x, s) = 0 \qquad \forall x \in P. \tag{17}$$

Since $(\Delta + k^2)u = 0$ in $R^3_+ = \{x : x_3 > 0\}$, $u(\infty) = 0$ and (17) holds, one concludes that $u = 0$ in R^3_+. By the unique continuation property for elliptic equations $u = 0$ in $R^3 \backslash D$. Thus $u = 0$ on Γ. Since $(\Delta + k^2)u = 0$ in D, $u = 0$ on Γ, and k^2 is not an eigenvalue of the Dirichlet Laplacian in D, one concludes that $u = 0$ in D. Therefore $h = 0$, being the jump of the normal derivative of u across Γ. Thus, the set $\{m(s)\}$, $s \in \Gamma$, is dense in $L^2(\Gamma)$ and the set $\{m(z)\}$ is dense in $N_D(\Delta + k^2)$.

In a similar way one proves that the set $\{n(z)\}$ is dense in $N_D(\Delta + k^2 - a(x))$. Lemma 2 is proved.

Lemma 3. The knowledge of integrals (15) determines $a(z)$ uniquely.

Proof. We need two facts which are proved in [3]. The first is an algebraic lemma: there exist α, $\beta \in \mathbb{C}^3$ such that $\alpha \cdot \alpha = \beta \cdot \beta = k^2$, $\alpha + \beta = p$, $|\alpha| \to \infty$, $|\beta| \to \infty$. Here $\alpha \cdot \beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$, $|\alpha| = (\alpha \cdot \bar{\alpha})^{1/2}$, where the bar stands for complex conjugate, and $p \in R^3$ is an arbitrary given vector. The second fact is an analytic lemma: there is a solution to the equation

$$\Delta w + k^2 w - a(x)w = 0 \qquad a(x) \in L^\infty(D) \tag{18}$$

of the form

$$w(x, \alpha) = \exp(\mathrm{i}\alpha \cdot x)(1 + R(x, \alpha)) \qquad \alpha \cdot \alpha = k^2, \quad \alpha \in \mathbb{C}^3 \tag{19}$$

where

$$\|R(x, \alpha)\|_{L^2(D_1)} \leqslant c(1 + |\alpha|)^{-1/2} \qquad \mathrm{Im}\, \alpha \neq 0, \quad |\alpha| \to \infty. \tag{20}$$

Here D_1 is an arbitrary bounded domain in R^3, $c = c(D_1) = \text{constant} > 0$ and c does not depend on α. Using these two facts, take in (15) $m = \exp(i\alpha \cdot z)$, $\alpha \cdot \alpha = k^2$ and $n = \exp(i\beta \cdot z)(1 + R(z, \beta))$, $\beta \cdot \beta = 0$, $\alpha + \beta = p$, $|\beta| \to \infty$. Passing to the limit $|\alpha| \to \infty$, $|\beta| \to \infty$ in (14) and using (20) one obtains on the left-hand side of (14) the integral

$$\int a(z) \exp(ip \cdot z)\mathrm{d}z := A(p). \tag{21}$$

Since the right-hand side of (14) is known, one knows the Fourier transform of $a(z)$. Thus, $a(z)$ is uniquely recovered.

This completes the proof of lemma 3. The theorem is proved.

Remark. Knowledge of $u(x, y, k)$ for all $x, y \in \mathrm{P}$ and a single $k > 0$ is not sufficient for the recovery of both a_j, $j = 1, 2$, because knowledge of $-q(x) + k^2 p(x)$ is not sufficient for the recovery of both $q(x)$ and $p(x)$.

In [7] the results of the theorem are proved for $a_1 \in L^2(D)$, $a_2 \in W^{2,2}(D)$, $1 + a_2 > 0$.

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