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LETTER TO THE EDITOR

Completeness of the products of solutions to PDE and uniqueness theorems in inverse scattering

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Abstract. Let $N=\{u: (\nabla^2+k^2)u=0\}$, $N_1=\{\psi: (\nabla^2+k^2-q(x))\psi=0\}$ in $\mathscr{D}\subset R^3$, where $\mathscr{D}$ is a bounded domain, $k=\text{constant}>0$, $q(x)\in L^\infty(\mathscr{D})$. Suppose that $f\in L^p(\mathscr{D})$, $p\geqslant 1$, and $\int f u\psi\,dx=0$ for all $u\in N$ and $\psi\in N_1$. Then $f=0$. Results of this type are used to prove uniqueness theorems in inverse scattering. In particular, we prove that the scattering amplitude $A(\theta',\theta,k)$ known at a fixed $k>0$ for all $\theta',\theta\in S^2$ determines the compactly supported potential $q(x)$ uniquely. We also prove that the surface data $u(x,y,k)$, $\forall\, x,y\in P=\{x: x_3=0\}$ at a fixed $k>0$ determine the compactly supported $v(x)\in L^2(\mathscr{D})$ uniquely. Here $[\nabla^2+k^2+k^2v(x)]=-\delta(x-y)$ in R^3, $\mathscr{D}\subset R^3_-=\{x: x_3<0\}$ is a bounded domain.

In [1] the following result is obtained. Let $Lu=\Sigma_{|\alpha|\leqslant l}\, a_\alpha D^\alpha u(x)$, $x\in R^d$, $d\geqslant 2$, $a_\alpha=$ constant, α is a multi-index. Let $M=\{z: z\in\mathbb{C}^d, \Sigma_{|\alpha|\leqslant l} a_\alpha z^\alpha=0\}$. Assume that there exist two points $m_j\in M$, $j=1,2$, such that the tangent planes T_j to M at points m_j are not parallel, and that

$$\int_{\mathscr{D}} f(x)u(x)w(x)\,dx=0 \qquad \forall\, u,w: u,w\in N_{\mathscr{D}}(L)=\{u: Lu=0 \text{ in } \mathscr{D}\} \quad (1)$$

where $f\in L^p(\mathscr{D}), p\geqslant 1$ and $\mathscr{D}$ is a bounded domain in R^d.

Theorem 0. Under the above assumptions $f=0$.

A generalisation of this result, also given in [1], says that the conclusion $f\equiv 0$ holds if (1) holds for all $u\in N(L)$ and $w\in N(L_1)$ where $L_1w=\Sigma_{|\alpha|\leqslant l_1} b_\alpha D^\alpha w$, and the union of the basis vectors in T_m, $m\in M$, and in T_{m_1}, $m\in M_1$ (where M_1 is defined by L_1 in the same way as M is defined by L) spans an open set in R^d. Sufficient conditions are given in [1] for this to be true in terms of the rank of the union of the basis vectors in T_m and T_{m_1}.

We first show how to generalise these results for some operators with variable coefficients. Consider, for example, $lu=[\nabla^2+k^2-q(x)]u$, $q(x)\in L^\infty_{\text{loc}}(R^d)$.

Theorem 1. If $\mathscr{D}$ is a bounded domain and (1) holds for all $u,w\in N_{\mathscr{D}}(l)$ then $f=0$.

Proof. Let (1) hold for all $u,w\in N_{\mathscr{D}}(l)$. By proposition 1 (at the end of the paper) one has

$$\int_{\mathscr{D}} f\exp[i(z+\zeta)\cdot x](1+o(1))\,dx=0 \qquad |z|\to\infty \qquad |\zeta|\to\infty \qquad (2)$$

where $z \cdot z = k^2 = \zeta \cdot \zeta$. Let us show that one can choose $z + \zeta = p \in R^3$, where p runs through a ball $B: |p| \leqslant R$, $R = \text{constant} > 0$. If this is established then (2) implies $\int_{\mathscr{D}} f \exp(ip \cdot x)\, dx = 0$, $p \in B$. Since the integral is an entire function of p which vanishes in B, it vanishes identically. Therefore $f \equiv 0$.

Let us check now that for any $p \in B$ there exist z and $\zeta \in \mathbb{C}^3$ such that $|\zeta| \to \infty$, $|z| \to \infty$, $z \cdot z = \zeta \cdot \zeta = k^2 > 0$. Let $z = a + ib$, $\zeta = \alpha + i\beta$. Then $a^2 - b^2 = \alpha^2 - \beta^2 = k^2$, $a \cdot b = \alpha \cdot \beta = 0$, $\alpha_j + a_j = p_j$, $b_j = \beta_j$, $j = 1, 2, 3$. We have ten equations for 12 coordinates. Let us show that these equations can be satisfied and $|z| \to \infty$, $|\zeta| \to \infty$. Eliminate a and b to get $\alpha^2 - \beta^2 = k^2$, $\alpha \cdot \beta = 0$, $(p - \alpha)^2 - b^2 = p^2 - 2p \cdot \alpha + \alpha^2 - \beta^2 = k^2$ thus $p^2 = 2p \cdot \alpha$, $(p - \alpha) \cdot \beta = 0$ thus $p \cdot \beta = 0$. Choose the third coordinate axis along p, i.e. take $p = \gamma e_3$, and take $\beta = \beta_1 e_1$, $\alpha = \alpha_3 e_3 + \alpha_2 e_2$.

Then $p \cdot \beta = 0$, $\alpha \cdot \beta = 0$, $\alpha_3^2 + \alpha_2^2 - \beta_1^2 = k^2$, $\gamma^2 = 2\gamma\alpha_3$, i.e. $\alpha_3 = \frac{1}{2}\gamma$. Thus $\alpha_2^2 - \beta_1^2 = k^2 - \frac{1}{4}\gamma^2$ (*). Take $|\alpha_2| \to \infty$, $|\beta_1| \to \infty$ so that (*) holds. This is possible. Set $\beta = \beta_1 e_1$, $\alpha = \alpha_2 e_2 + \frac{1}{2}\gamma e_3$, $b = \beta$, $a = \gamma e_3 - \alpha$. Then all ten equations are satisfied. Theorem 1 is proved.

Let $q \in L^\infty(\mathscr{D})$, $\mathscr{D} \subset R^3$ is a bounded domain, $A(\theta', \theta, k)$ is the scattering amplitude corresponding to $q(x)$ (see e.g. [3]),

$$A(\theta', \theta, k) = -(4\pi)^{-1} \int dx\, q(x) \exp(-ik\theta' \cdot x)\psi(x, \theta, k) \qquad \int = \int_{R^3} \tag{3}$$

where ψ solves the scattering problem: $[\nabla^2 + k^2 - q(x)]\psi = 0$ in R^3, $\psi = \exp(ik\theta \cdot x) + v$, $\theta \in S^2$, the unit sphere in R^3, $v = g(r)A(\theta', \theta, k) + o(r^{-1})$ as $r = |x| \to \infty$, $\theta' = xr^{-1}$, $g = r^{-1} \exp(ikr)$.

Theorem 2. Let $q(x) \in L^\infty(\mathscr{D})$. If $A(\theta', \theta, k)$ is known at a fixed $k > 0$ for all θ', $\theta \in S^2$ then $q(x)$ is uniquely determined.

Proof. Apply proposition 1 to (3): take $\psi = \exp(ik\theta \cdot x)(1 + r(x, \theta))$, $\|r(x, \theta)\|_{L^2(\mathscr{D})} = o(1)$ as $|\theta| \to \infty$, $\theta \in \mathbb{C}^3$, $\theta \cdot \theta = 1$, $\theta' \cdot \theta' = 1$, $\theta' \in \mathbb{C}^3$. The function $A(\theta', \theta, k)$ is analytic in θ', θ on the subset of $\mathbb{C}^3 \times \mathbb{C}^3$ given by the equations $\theta' \cdot \theta' = 1$, $\theta \cdot \theta = 1 (\cdot\cdot)$. The values of $A(\theta', \theta, k)$ for real θ', θ define this function on the set $(\cdot\cdot)$ uniquely by analytic continuation. For example, if one takes $\theta' = (\theta_1', \theta_2', (1 - \theta_1'^2 - \theta_2'^2)^{1/2})$ then A is analytic in (θ_1', θ_2') in a neighbourhood of the origin in $\mathbb{C}^2$ and is known on R^2. By analytic continuation it is uniquely determined in a neighbourhood of the origin in $\mathbb{C}^2$. This means that $A(\theta', \theta, k)$ is determined on the set in $\mathbb{C}^3$ given by equations $(\cdot\cdot)$. We wish to show that one can choose $\theta, \theta' \in \mathbb{C}^3$ on the set $(\cdot\cdot)$ so that: $i_1) |\theta'| \to \infty$ and $i_2)$ $k(\theta - \theta') = p$, where $p \subset B \subset R^3$ and B is a ball. Without loss of generality take $k = 1$ and $p = \gamma e_3$. Then

$$\begin{aligned} &\theta_j = \theta_j' + \gamma e_3 \qquad \theta := a + ib \qquad \theta' := \alpha + i\beta \qquad a, b, \alpha, \beta \in R^3 \\ &a^2 - b^2 = \alpha^2 - \beta^2 = 1 \qquad a \cdot b = \alpha \cdot \beta = 0. \end{aligned} \tag{4}$$

One has $b = \beta$, $a_1 = \alpha_1$, $a_2 = \alpha_2$, $a_3 = \alpha_3 + \gamma$. One has ten equations for 12 coordinates, namely: $\theta \cdot \theta = \theta' \cdot \theta' = 1$ and $\theta \cdot \theta' = p$ which is $i_2)$ for $k = 1$. Let us show that these equations can be satisfied under condition i_1). Eliminate a and b to get

$$a^2 - b^2 = \alpha_1^2 + \alpha_2^2 + (\alpha_3 + \gamma)^2 - \beta^2 = 1 \Rightarrow 2\alpha_3\gamma + \gamma^2 = 0 \Rightarrow \alpha_3 = -\tfrac{1}{2}\gamma;$$

$$0 = a \cdot b = \alpha_1\beta_1 + \alpha_2\beta_2 + (\alpha_3 + \gamma)\beta_3 = \gamma\beta_3 \Rightarrow \beta_3 = 0,$$

since $\gamma\neq 0$. One has two more equations $\alpha\cdot\beta=\alpha_1\beta_1+\alpha_2\beta_2=0$ and $1=\alpha^2-\beta^2=\alpha_1^2+\alpha_2^2+\frac{1}{4}\gamma^2-\beta_1^2-\beta_2^2$. Take $\alpha_1=c_1$, $\alpha_2=0$, $\beta_1=0$, $\beta_2=c_2$. Then $c_1^2+\frac{1}{4}\gamma^2-c_2^2=1$. This equation can be satisfied and $|c_2|$ can be taken to infinity so that i_1) holds. Therefore $\tilde{q}(p)$ is uniquely defined by the values of $A(\theta', \theta, \mathrm{k})$ at a fixed $\mathrm{k}>0$ for all $\theta', \theta\in S^2$. Thus q is uniquely determined. Theorem 2 is proved. The scattering solution ψ is representable in the form (28)–(29) as follows from the integral equation for ψ.

Remark 1. It is sufficient to give $A(\theta', \theta, \mathrm{k})$ at a fixed $\mathrm{k}>0$ for $\theta', \theta\in\tilde{S}^2\times\tilde{S}^2$, where $\tilde{S}^2$ is a solid angle (that is, an open set in S^2). By analyticity these data determine A on all of $S^2\times S^2$ uniquely ([2], p 62).

Next consider the equation

$$[\nabla^2+k^2+k^2v(x)]u(x, y, k)=-\delta(x-y) \qquad \text{in } R^3 \tag{5}$$

Assume that $v\in L^2(\mathscr{D})$, $\mathscr{D}\subset R^3_-=\{x: x_3<0\}$, $\mathscr{D}$ is bounded, and the surface data $\mathrm{u}(x, y, \mathrm{k})$ are given at a fixed $k>0$ for all $x, y\in P=\{x: x_3=0\}$.

Theorem 3. The above data determine $v(x)$ uniquely.

Remark 2. In [2] it is proved that the surface data known for all $0<k<k_0$, where $k_0>0$ is arbitrarily small, determine $v(x)$ uniquely.

Proof of theorem 3. The solution to (5) which satisfies the radiation condition solves the equation:

$$u_s(x, y):=u-g=\int g(x, \xi)v(\xi)u(\xi, y)\,\mathrm{d}\xi \qquad g=(4\pi|x-y|)^{-1}\exp(\mathrm{i}k|x-y|) \tag{6}$$

where u_s is the scattered field. The integral in (6) is known for all $x, y\in P$. We wish to prove that these data determine $v(\xi)$ uniquely.

Step 1. If $u_s(x, y)$ is known for all $x, y\in P$ then the numbers

$$\int_{\mathscr{D}} \mathrm{d}\xi\, v(\xi)u_m(\xi)w_{m'}(\xi):=v_{mm'} \text{ are known for all}$$
$$u_m\in N_{\mathscr{D}}(\nabla^2+k^2) \qquad \text{and} \qquad w_{m'}\in N_{\mathscr{D}}(\nabla^2+k^2+k^2v(x)). \tag{7}$$

Step 2. Use proposition 1 to conclude that if $v_{mm'}$ are known then $v^0_{mm'}$ are known, $v^0_{mm'}:=\int_{\mathscr{D}} d\xi\, v(\xi)u_m(\xi)u_{m'}(\xi)$, and $w_{m'}(x)=\exp(\mathrm{i}z\cdot x)(1+\mathrm{o}(1))$ as $|z|\to\infty$, $z\cdot z=k^2$. If the numbers $v^0_{mm'}$ are known then v is uniquely determined as in theorem 0.

Let us explain step 1 in more detail. Multiply (6) by a $\varphi\in C_0^\infty(P)$ and integrate in x to get:

$$F(\varphi, y):=\int h(\xi)v(\xi)u(\xi, y)\,\mathrm{d}\xi \tag{8}$$

F is known for all $y\in P$ and all $\varphi\in C_0^\infty(P)$. Here

$$h(\xi):=\int_P g(x, \xi)\varphi\,\mathrm{d}x \qquad (\nabla^2+k^2)h=0 \text{ in } \mathscr{D}. \tag{9}$$

In order to show that h runs through a dense set in $N_{\mathscr{D}}(\nabla^2+k^2)$ when φ runs through $C_0^\infty(P)$ assume $(*)$: k^2 is not an eigenvalue of the Dirichlet operator ∇^2+k^2 in $\mathscr{D}$ (if k^2 is an eigenvalue then take $\mathscr{D}_1\supset\mathscr{D}$, $\mathscr{D}_1\subset R^3_-$, such that k^2 is not an eigenvalue of the

Dirichlet operator ∇^2+k^2 in $\mathscr{D}_1$ and use the argument below). We will show that $h|_{\partial\mathscr{D}}$ runs through a dense subset of $L^2(\partial\mathscr{D})$. This implies that h runs through a dense set in $N_{\mathscr{D}}(\nabla^2+k^2)$ provided that $(*)$ holds. To prove that the set $\{h\}$ is dense in $L^2(\partial\mathscr{D})$ assume that there is a function $\eta(x)\in L^2(\partial\mathscr{D})$ such that $\int_{\partial\mathscr{D}}\eta h\,\mathrm{d}s=0$ for all h of the form (9). Then

$$\begin{aligned} w(x):=\int_{\partial\mathscr{D}} g(s,x)\eta(s)\,\mathrm{d}s=0 \qquad \forall\, x\in P \\ (\nabla^2+k^2)w=0 \qquad \text{in} \qquad R^3\backslash\partial\mathscr{D}. \end{aligned} \tag{10}$$

This implies that $w=0$ outside of $\mathscr{D}$, $w=0$ on $\partial\mathscr{D}$, and, since $(*)$ holds, $w=0$ in $\mathscr{D}$. By the jump relation for $[\partial w/\partial N]_{\partial\mathscr{D}}$ one has $\eta=0$. Therefore the set $\{h\}$ is dense in $L^2(\partial\mathscr{D})$. In a similar way one proves that (6) implies (7).

Let us prove proposition 1 used above. First we need two lemmas.

Lemma 1. If $|z|$ is *sufficiently large* and $f\in L^2(\mathscr{D})$ where $\mathscr{D}\subset R^3$ is a *bounded* domain, then the equation

$$Lu:=(\nabla^2+\mathrm{i}z\cdot\nabla)u=f \qquad z\in\mathbb{C}^3,\quad z\cdot z=k^2=\text{constant}\geqslant 0,\quad \operatorname{Im} z\neq 0 \tag{11}$$

is solvable in $L^2_{\mathrm{loc}}(R^3)$ and

$$\|u\|_{L^2(\mathscr{D})}\leqslant c|z|^{-1/2}\|f\|_{L^2(\mathscr{D})} \qquad |z|\gg 1,\quad \operatorname{Im} z\neq 0,\quad z\cdot z=k^2. \tag{12}$$

Here and below c denote various constants. The constant in (12) depends on the domain $\mathscr{D}$.

Proof. Let $\tilde{u}:=(2\pi)^{-3}\int\exp(-\mathrm{i}\lambda\cdot x)u(x)\,\mathrm{d}x$, $\int=\int_{R^3}$. Then (11) has a distribution solution $u(x)=-\int\tilde{f}(\lambda)(\lambda^2+z\cdot\lambda)^{-1}\exp(\mathrm{i}\lambda\cdot x)\,\mathrm{d}\lambda$. Let $z=a+\mathrm{i}b$, $a,b\in R^3$. Then $z\cdot z=a^2-b^2+2\mathrm{i}a\cdot b=k^2>0$. Thus

$$a^2-b^2=k^2 \qquad a\cdot b=0. \tag{13}$$

Let $a=e_3t_k$, $b=e_1t$, where e_j, $1\leqslant j\leqslant 3$, are the orthonormal basis vectors in R^3, $t=|b|>0$, $t_k=|a|=(t^2+k^2)^{1/2}>0$. Then

$$\lambda^2+z\cdot\lambda=\lambda_1^2+\mathrm{i}\lambda_1t+\lambda_2^2+\lambda_3^2+t_k\lambda_2=\lambda_1^2+\mathrm{i}\lambda_1t+\lambda_3^2+(\lambda_2+\tfrac{1}{2}t_k)^2-\tfrac{1}{4}t_k^2. \tag{14}$$

Function (14) vanishes iff

$$\lambda_1=0 \qquad \text{and} \qquad \lambda_3^2+(\lambda_2+\tfrac{1}{2}t_k)^2-\tfrac{1}{4}t_k^2=0. \tag{15}$$

Equation (15) defines a circle C_k in the plane (λ_2,λ_3) centred at $(0,-\tfrac{1}{2}t_k,0)$ with radius $\tfrac{1}{2}t_k$. Consider the torus T_δ which is obtained by moving a square with the side 2δ centred at the points of C_k and perpendicular to the plane (λ_2,λ_3). One has

$$\begin{aligned} |u(x)|\leqslant{}&\left|\int_{T_\delta}\tilde{f}(\lambda)(\lambda^2+z\cdot\lambda)^{-1}\exp(\mathrm{i}\lambda\cdot x)\,\mathrm{d}\lambda\right| \\ &+\left|\int_{R^3\backslash T_\delta}\tilde{f}(\lambda)(\lambda^2+z\cdot\lambda)^{-1}\exp(\mathrm{i}\lambda\cdot x)\,\mathrm{d}\lambda\right|:=|v|+|w|. \end{aligned} \tag{16}$$

Clearly $|\lambda^2+z\cdot\lambda|\geqslant\tfrac{1}{2}\delta t$ if $\lambda\notin T_\delta$. Therefore, by Parseval's equality,

$$\|w\|_{L^2(R^3)}=\left(\int_{R^3\backslash T_\delta}|\tilde{f}|^2|\lambda^2+z\cdot\lambda|^{-2}\,\mathrm{d}\lambda\right)^{1/2}\leqslant 2(\delta t)^{-1}\|f\|_{L^2(\mathscr{D})}. \tag{17}$$

Let us estimate $v(x)$:

$$|v(x)| \leqslant 2\pi \int_{-\delta}^{\delta} d\lambda_1 \int_{t_k/2-\delta}^{t_k/2+\delta} \frac{dr\, r}{[\lambda_1^2 t^2 + (\lambda_1^2 + r^2 - \frac{1}{4}t_k^2)^2]^{1/2}} \|\tilde{f}\|_{L^\infty(R^3)}$$

$$\leqslant 8\pi \|f\|_{L^1(\mathscr{D})} \int_0^{\delta} d\lambda_1 \int_{\lambda_1^2 - t_k\delta + \delta^2}^{\lambda_1^2 + t_k\delta + \delta^2} \frac{d\mu}{(\lambda_1^2 t^2 + \mu^2)^{1/2}} \leqslant c\|f\|_{L^2(\mathscr{D})}.$$

$$\times \int_0^{\delta} d\lambda_1 \int_{-2t_k\delta}^{2t_k\delta} d\mu (\lambda_1^2 t^2 + \mu^2)^{-1/2} \tag{18}$$

where $\lambda_1^2 + r^2 - \frac{1}{4}t_k^2 = \mu$, $d\mu = 2r\, dr$, we used the inequalities $\lambda_1^2 \leqslant \delta^2$, $\lambda_1^2 + \delta^2 \leqslant t_k\delta$ which hold if $0 < \delta < 1$ and $t_k \geqslant 2$, and we assumed that the last two inequalities are valid.

Let $\lambda_1 t = \beta$. Then the integral in (18) is not greater than

$$2t^{-1} \int_0^{\delta t} \int_0^{2\delta t_k} d\beta\, d\mu (\beta^2 + \mu^2)^{-1/2} \leqslant c_1 t^{-1} \int_0^{2\delta t_k} d\rho \leqslant c_2 t^{-1} t_k \delta \leqslant c\delta \tag{19}$$

where $c = \text{constant} > 0$, polar coordinates were used and we took into account that $t^{-1}t_k = t^{-1}(t^2 + k^2)^{1/2} \leqslant 2$ for sufficiently large t and a fixed $k > 0$. Combining (18) and (19) one gets

$$\|v(x)\|_{L^\infty(R^3)} \leqslant c\|f\|_{L^2(\mathscr{D})}\delta. \tag{20}$$

Thus

$$\|v\|_{L^2(\mathscr{D})} \leqslant c\|f\|_{L^2(\mathscr{D})}\delta. \tag{21}$$

Combining (16), (17) and (21) one obtains

$$\|u\|_{L^2(\mathscr{D})} \leqslant c\|f\|_{L^2(\mathscr{D})}\left(\frac{1}{\delta t} + \delta\right). \tag{22}$$

Note that one can put any bounded domain $\mathscr{D}_1 \supset \mathscr{D}$ in place of $\mathscr{D}$ in (22). For a fixed $t > 0$ one has $\min_\delta[(\delta t)^{-1} + \delta] = 2t^{-1/2}$. Therefore

$$\|u\|_{L^2(\mathscr{D})} \leqslant c\|f\|_{L^2(\mathscr{D})} t^{-1/2}. \tag{23}$$

This is equivalent to (12). Therefore the existence of a solution to (1) with the properties: (i) $u \in L^2_{\text{loc}}(R^3)$, and (ii) u satisfies the estimate (12), is proved. Uniqueness of this solution follows from inequality (12) since equation (11) is linear.

Lemma 2. Let $q(x) \in L^\infty(\mathscr{D})$, $q = 0$ if $x \notin \mathscr{D}$. Consider the equation

$$[\nabla^2 + iz \cdot \nabla - q(x)]u = f \qquad \text{in } R^3. \tag{24}$$

Under the assumptions of lemma 1 equation (24) is solvable in $L^2_{\text{loc}}(R^3)$ and the inequality (12) holds for the solution to (24).

Proof Let us write (24) as

$$Lu = qu + f. \tag{25}$$

If $u \in L^2_{\text{loc}}$ solves (15) then $qu \in L^2(R^3)$ since $q \in L^\infty(\mathscr{D})$ and is compactly supported. By lemma 1 one has

$$\|u\|_{L^2(\mathscr{D})} \leqslant c|z|^{-1/2}(\|f\|_{L^2(\mathscr{D})} + \|qu\|_{L^2(\mathscr{D})}) \leqslant c|z|^{-1/2}\|f\|_{L^2(\mathscr{D})} + c|z|^{-1/2}\|q\|_{L^\infty(\mathscr{D})}\|u\|_{L^2(\mathscr{D})}.$$

Therefore u satisfies the estimate (12). Let us prove that equation (25) is solvable in L^2_{loc} provided that $|z|$ is sufficiently large. For any compact domain $\mathscr{D}_1$ and $u \in L^2(\mathscr{D}_1)$ one has

$$\|L^{-1}qu\|_{L^2(\mathscr{D}_1)} \leqslant c|z|^{-1/2}\|q\|_{L^\infty(R^3)}\|u\|_{L^2(\mathscr{D}_1)}. \tag{26}$$

Therefore (25) can be written as $u = L^{-1}qu + L^{-1}f$ and if $|z|$ is sufficiently large this equation is uniquely solvable in $L^2(\mathscr{D}_1)$ by the contraction mapping principle: according to estimate (26) the operator $L^{-1}q$ is a contractive mapping in $L^2(\mathscr{D}_1)$ if $|z| \gg 1$. Lemma 2 is proved.

We can now prove

Proposition 1. Let $q(x) \in L^\infty(\mathscr{D})$, $\mathscr{D} \subset R^3$ is a bounded domain, $q(x) = 0$ for $x \notin \mathscr{D}$. Then the equation

$$[\nabla^2 + k^2 - q(x)]u = 0 \tag{27}$$

for any fixed $k \geqslant 0$ has a solution of the form

$$u = \exp(\mathrm{i}z \cdot x)(1 + r(x, z)), \tag{28}$$

where $z \in \mathbb{C}^3$, $z \cdot z = k^2$ and

$$\|r(x, z)\|_{L^2(\mathscr{D}_1)} = 0(|z|^{-1/2}) \qquad \text{as } |z| \to \infty \qquad \operatorname{Im} z \neq 0, \tag{29}$$

uniformly in $\mathscr{D}_1$ running through any bounded domain in R^3. If $q \in L^{m,\infty}(\mathscr{D})$ then $\|r(x, z)\|_{H^m(\mathscr{D}_1)} = 0(|z|^{-1/2})$ as $|z| \to \infty$.

Proof. Substitute (28) into (27) and use $z \cdot z = k^2$ to get

$$\nabla^2 r + 2\mathrm{i}z \cdot \nabla r - q(x)r = q(x) \tag{30}$$

Apply lemma 2 to equation (30) to get the conclusion of proposition 1.

In a very interesting work [3] a similar result is given for $k = 0$. The argument in [3] is longer and more complicated.

Remark 3. The conclusions of lemma 2 and proposition 1 remain valid if $q \in L^3(\mathscr{D})$. The estimates (12) and (29) hold with $|z|^{-(1-\varepsilon)}$ in place of $|z|^{-1/2}$, $\varepsilon > 0$ is arbitrarily small, and with $\|u\|_{L^\infty(\mathscr{D}_1)}$, $\|r(x, z)\|_{L^\infty(\mathscr{D}_1)}$. In [3] estimate (29) with $O(|z|^{-1})$ is proved for $k = 0$.

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