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LETTER TO THE EDITOR

Necessary and sufficient conditions on the scattering data for the potential to be in L^2 for the Schrödinger operator on the half-line

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Abstract. Necessary and sufficient conditions for a potential $q(x)$ to be in L^2 are given in terms of $S(k)$ or $f(k)$. Necessary and sufficient condition for q to be compactly supported and in L^2 are given in terms of $f(k)$. Here $f(k)$ is the Jost functions and $S(k)$ is the scattering matrix of the Schrödinger operator on the half-line.

Let

$$f''+k^2f-q(x)f=0 \qquad x>0, \quad q(x)=\bar{q}(x), \quad \int_0^{\infty}(1+x)|q|\,\mathrm{d}x<\infty \tag{1}$$

$$f(x,k)=f_0+\mathrm{o}(1) \qquad \text{as } x\to+\infty, \quad f_0=\exp(\mathrm{i}kx) \quad k>0. \tag{2}$$

The bar denotes complex conjugate. The function $f(0,k):=f(k)$ is the Jost function $S(k)=f(-k)/f(k)$ is the S matrix. The inverse scattering problem consists in finding $q(x)$ from the scattering data $\{S(k),\lambda_j,M_j\}$, $1\leqslant j\leqslant n$, where λ_j and M_j are given positive constants. Necessary and sufficient conditions were found for the data $\{S(k),\lambda_j,M_j\}$, to be the scattering data corresponding to a $q\in A:=\{q\colon q=\bar{q}, \int_0^{\infty}(1+x)|q|\,\mathrm{d}x<\infty\}$ [1]. It was not known under what conditions this q belongs to $L^2=L^2(0,\infty)$. Our first result is

Theorem 1. Suppose $q\in A$. Then $q\in L^2$ iff one of the following conditions holds:

$$k[|f(k)|^2-1]\in L^2(-\infty,\infty) \tag{3}$$

$$k[f(k)-1+(2\mathrm{i}k)^{-1}Q]\in L^2(-\infty,\infty) \tag{4}$$

$$k[1-S(k)-\mathrm{i}Qk^{-1}]\in L^2(-\infty,\infty) \tag{5}$$

where $Q=\int_0^{\infty} q(x)\,\mathrm{d}x$ can be computed by the formulae

$$Q=-\mathrm{i}\lim_{k\to\infty}[k(1-S(k))] \qquad Q=\lim_{k\to\infty}[-2\mathrm{i}k(f(k)-1)]. \tag{6}$$

Our second result is

Theorem 2. Assume that $q\in A$. Then $q\in L^2$ and $q=0$ for $x\geqslant R$ iff $f(k)$ is a restriction to the real axis of an entire function of complex variable k of exponential type $\leqslant 2R$, and condition (3) (or (4)) holds.

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Remark 1. It is known that $f(k)\neq 0$ for real $k\neq 0$, and that $f(0)=(0)$ may happen [1]. In the proof of theorem 2 we use the condition $f(k)\neq 0$ for $\operatorname{Im} k\geqslant 0$. For simplicity the proof of theorem 2 is given under the assumption $f(0)\neq 0$. If $f(0)=0$ then one can still use the argument given in the proof of theorem 2 below formula (20), but the reference to the Wiener–Levy theorem should be substituted by the statement that $(1+\tilde{A}_+(-\lambda))^{-1}=1+B_-(\lambda)$ where $B_-(\lambda)$ is the Fourier transform of a function $b_-(t)$; $b_-(t)=0$ for $t>0$ but $b_-(t)$ is not necessarily in L^1.

Remark 2. If $\operatorname{ind} S(k)=0$, where $\operatorname{ind} S$ is the index of $S(k)$ on the contour $(-\infty, \infty)$, $\operatorname{ind} S:=(2\pi)^{-1}\Delta_{(-\infty,\infty)}\arg S(k)$, and where $\Delta_{(-\infty,\infty)}\arg S$ denotes the increment of the $\arg S(k)$ on the contour $(-\infty,\infty)$, then $f(k)\neq 0$ for $\operatorname{Im} k\geqslant 0$, in particular, for $-\infty<k<\infty$.

The result in theorem 2 refines an earlier result [2]. In [3, 5] some related results can be found.

Proof of theorem 1. From the known [1, 4] equations

$$f(k)=1+\int_0^\infty \mathrm{d}t\, k^{-1}\sin(kt)q(t)f(t,k) \tag{7}$$

$$f(t,k)=\exp(\mathrm{i}kt)+\int_t^\infty k^{-1}\sin[k(t'-t)]q(t')f(t',k)\,\mathrm{d}t \tag{8}$$

it follows that

$$\begin{aligned} f(k)={}&1+\int_0^\infty \mathrm{d}t\, k^{-1}\sin(kt)q(t)\exp(\mathrm{i}kt)\\ &+\int_0^\infty \mathrm{d}t\, k^{-1}\sin(kt)q(t)\int_t^\infty k^{-1}\sin[k(t'-t)]q(t')f(t',k)\,\mathrm{d}t\\ ={}&1+(2\mathrm{i}k)^{-1}\int_0^\infty \mathrm{d}t\, q(t)\exp(2\mathrm{i}kt)-(2\mathrm{i}k)^{-1}Q+\mathrm{O}(k^{-2}) \qquad k\to+\infty \end{aligned} \tag{9}$$

where $Q:=\int_0^\infty q\,\mathrm{d}t$. Use (9) and the definition of $S(k):=f(-k)/f(k)$ to get

$$\begin{aligned} 1-S(k)&=2\mathrm{i}f^{-1}(k)\operatorname{Im} f\\ &=\mathrm{i}k^{-1}Q-\mathrm{i}k^{-1}\int_0^\infty \cos(2kt)q(t)\,\mathrm{d}t+\mathrm{O}(k^{-2}) \qquad k\to+\infty \end{aligned} \tag{10}$$

and

$$|f(k)|^2-1=k^{-1}\int_0^\infty \sin(2kt)q(t)\,\mathrm{d}t+\mathrm{O}(k^{-2}) \qquad k\to+\infty. \tag{11}$$

From (9) and (10) one obtains (6). Since the terms $\mathrm{O}(k^{-2})$ in formulae (9)–(11) are locally continuous in k, the terms $k\mathrm{O}(k^{-2})$ belong to $L^2(-\infty,\infty)$. Therefore the conclusions of theorem 1 follow from the fact that the Fourier transforms

$$\mathscr{F}q := \int_0^\infty q(t)\exp(2ikt)\,dt$$

$$\mathscr{F}_S q := \int_0^\infty q(t)\sin(2kt)\,dt$$

$$\mathscr{F}_C q := \int_0^\infty q(t)\cos(2kt)\,dt$$

are in L^2 iff $q \in L^2$.

Proof of Theorem 2. Note that

$$f(k) = 1 + \int_0^\infty A(y)\exp(iky)\,dy \tag{12}$$

where $A(y) \in L^1$ if $q \in A$. The conditions for $q \in L^2$ (condition (3) or (4)) were discussed in the proof of theorem 1. Therefore it is sufficient to show that, under these conditions, $q = 0$ for $x \geqslant R$ iff $f(k)$ is a restriction to the real axis of an entire function of exponential type $\leqslant 2R$. If $q = 0$ for $x \geqslant R$ then $f(x, k) = f_0(x, k)$ for $x > R$. This follows from the equation (8). Therefore $A(x, y) = 0$ for $R < x \leqslant y$, where the transformation kernel $A(x, y)$ is defined by the equation

$$f(x, k) = \exp(ikx) + \int_x^\infty A(x, y)\exp(iky)\,dy. \tag{13}$$

From the Marchenko equation

$$A(x, y) + F(x + y) + \int_x^\infty F(t + y)A(x, t)\,dt = 0 \qquad y \geqslant x \geqslant 0 \tag{14}$$

it follows [1] that $F(x + y) = 0$ for $y \geqslant x \geqslant R$, where

$$F(x) := \sum_{j=1}^{N} s_j \exp(-\kappa_j x) + (2\pi)^{-1}\int_{-\infty}^{\infty} (1 - S(k))\exp(ikt)\,dk \qquad \kappa_j = \lambda_j^{1/2}$$

Thus $F(t) = 0$ for $t \geqslant 2R$. Consider the equation for $A(y) := A(0, y)$,

$$A(y) + F(y) + \int_0^\infty F(t + y)A(t)\,dt = 0 \qquad y \geqslant 0. \tag{15}$$

Since $F(y) = 0$ for $y \geqslant 2R$ it follows from (15) that

$$A(y) = 0 \qquad \text{for } y \geqslant 2R. \tag{16}$$

Therefore

$$f(k) = 1 + \int_0^{2R} A(y)\exp(iky)\,dy \tag{17}$$

is an entire function of exponential type $\leqslant 2R$. We have proved that $q \in A$ and $q = 0$ for $x \geqslant R$ implies that $f(k)$ is an entire function of exponential type $\leqslant 2R$.

Now let us prove the converse. Assume that $f(k)$ is an entire function of exponential type $\leqslant 2R$. Then (16) and (17) follow from (12) and the Paley–Wiener theorem.

Let us sketch the rest of the argument. From (16) and (15) we derive that $F(t)=0$ for $t \geqslant 2R$. This and (14) imply immediately that $A(x,y)=0$ for $y \geqslant x \geqslant R$. Since $q=-2\mathrm{d}A(x,x)/\mathrm{d}x$ it follows that $q(x)=0$ for $x \geqslant R$. This ends the argument.

Let us show that (15) and (16) imply $F=0$ for $t \geqslant 2R$. Write equation (15) as

$$A_+(y)+F_+(y)+\int_{-\infty}^{\infty} F_+(t+y)A_+(t)\,\mathrm{d}t=A_-(y) \qquad -\infty<y<\infty \tag{18}$$

where

$$A_+(y):=\begin{cases} A(y) & y>0 \\ 0 & y<0 \end{cases} \qquad \begin{aligned} &A_-(y)=0 \quad \text{for } y \geqslant 0 \text{ and is defined to be} \\ &\text{the left-hand side of (18) for } y<0. \end{aligned} \tag{19}$$

Take the Fourier transform of (18) to get $\tilde{A}_+(\lambda)+\tilde{F}_+(1+\tilde{A}_+(-\lambda))=\tilde{A}_-(\lambda)$. Thus

$$\tilde{F}_+(\lambda)=-\tilde{A}_+(\lambda)(I+\tilde{A}_+(-\lambda))^{-1}+\tilde{A}_-(\lambda)(I+\tilde{A}_+(-\lambda))^{-1}. \tag{20}$$

Since $f(k)\neq 0$ for $\operatorname{Im} k \geqslant 0$, one has $1+\tilde{A}_+(-\lambda)\neq 0$ for $\operatorname{Im}\lambda \leqslant 0$. By the Wiener–Levy theorem the function $(1+\tilde{A}_+(-\lambda))^{-1}=1+B_-(\lambda)$, where $B_-(\lambda)$ is the Fourier transform of a function $b_-(t)\in L^1$, $b_-(t)=0$ for $t>0$. Define the projection $P_+\tilde{F}:=\tilde{F}_+$, where

$$\tilde{F}(\lambda)=\int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda t)F(t)\,\mathrm{d}t$$

$$\tilde{F}_+(\lambda):=\int_{0}^{\infty} \exp(\mathrm{i}\lambda t)F(t)\,\mathrm{d}t.$$

Since $P_+[\tilde{A}_-(\lambda)(1+B_-(\lambda))]=0$ one obtains from (20) that

$$\tilde{F}_+(\lambda)=-\tilde{A}_+(\lambda)-P_+(\tilde{A}_+(\lambda)B_-(\lambda)). \tag{21}$$

By (16) it is sufficient to show that the inverse Fourier transform of $\tilde{g}:=P_+(\tilde{A}_+(\lambda)B_-(\lambda))$ vanishes for $t>2R$. One has

$$g(t):=h(t)\int_0^{2R} A_+(t')b_-(t-t')\,\mathrm{d}t' \qquad h(t):=\begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases} \tag{22}$$

Since $b_-(t)=0$ for $t>0$ one has $g(t)=0$ for $t>2R$. This ends the proof of theorem 2.

Let us give a proof of the fact that (15) and (16) imply that $F=0$ for $t>2R$ in the general case when bound states are present and $f(0)=0$. This proof does not depend on the Wiener–Levy theorem. Without loss of generality assume that there is just one bound state, $-\kappa^2$. It is known that in the presence of one bound state

$$F(t)=s\exp(-\kappa t)+(2\pi)^{-1}\int_{-\infty}^{\infty}(1-f(-k)f^{-1}(k))\exp(\mathrm{i}kt)\,\mathrm{d}k \tag{23}$$

where, if $f(\kappa)$ is entire, the normalising constant is given by the formula

$$s:=\mathrm{i}f(-\mathrm{i}\kappa)[\dot{f}(\mathrm{i}\kappa)]^{-1} \qquad \dot{f}(k):=\mathrm{d}f/\mathrm{d}k. \tag{24}$$

If (16) holds then $f(k)$ is given by (17) and is an entire function of k. If $t>2R$ then the integral in (23) can be computed by the residue theorem applied to the contour $(-\infty,\infty)$

closed by the large semicircle in the upper half-plane of the complex plane k. Since zero is a regular point of the integrand in (23), the integral is equal to

$$-\mathrm{i}\operatorname*{Res}_{k=\mathrm{i}\kappa}[f(-k)f^{-1}(k)\exp(\mathrm{i}kt)]=-\mathrm{i}f(-\mathrm{i}\kappa)[\dot{f}(\mathrm{i}\kappa)]^{-1}\exp(-\kappa t)=-s\exp(-\kappa t) \qquad (25)$$

where we used the known fact that $f(k)$ has only simple zeros in the half-plane $\operatorname{Im} k>0$. From (25) and (23) it follows that $F(t)=0$ for $t>2R$ as claimed. If there are N bound states then $\Sigma_{j=1}^{N} s_j \exp(-\kappa_j t)$ should be in the place of the first term on the right-hand side of (23) and on the right-hand side of (25).

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