Inverse Problems 3 (1987) L59–L61. Printed in the UK

LETTER TO THE EDITOR

An inverse problem for Helmholtz's equation II

A G Ramm
Mathematics Department, Kansas State University, Manhattan, KS 66506, USA

Received 23 April 1987

Abstract. A method is given for recovery of $n(z)$ and $h(k)$ in

$$[\nabla^2 + k^2 n(z)]u = -\delta(x)h(k) \qquad \text{in } R^3$$

from the surface data.

Let

$$[\nabla^2 + k^2 n(z)]u = -\delta(x)h(k) \qquad \text{in } R^3 \tag{1}$$

where $x = (x_1, x_2, z)$, the refraction coefficient $n(z) = 1$ for $z > 0$ and is unknown for $z < 0$, $\delta(x)$ is the delta function describing the point source of acoustic waves and $h(k)$ is an unknown function that describes the frequency dependence of the wave source (or the shape of the wavelet in the time domain).

The problem. Given $u(x_1, x_2, 0, k)$ for all $-\infty < x_1, x_2 < \infty$ and all $k > 0$, find $n(z)$ for $z < 0$ and $h(k)$ for $k > 0$.

We assume that $h(-k) = \overline{h(k)}$, $u(x, -k) = \overline{u(x, k)}$, where the bar stands for the complex conjugate, and that the unknown function $n(z)$ for $z < 0$ satisfies the following conditions

$$n(z) \in L^2_{\text{loc}} \qquad |n(z)| \leqslant C(1 + |z|^m) \qquad \text{for } z \to -\infty \tag{2}$$

where $m \geqslant 0$ is a constant and C stands for various positive constants. Let $\tilde{L}_1$ denote the set of functions $h(k)$ that are the Fourier transforms of real-valued functions $H(t) \in L_1(0, \infty)$. Let us assume that (see Remark below)

$$h(k) \in \tilde{L}_1 \qquad h(0) \neq 0. \tag{3}$$

Let us describe a method for solving the problem. The method is based on the ideas and results in [1, 2].

Let us Fourier transform (1) in (x_1, x_2) to get

$$\tilde{u}'' - \lambda^2 \tilde{u} + k^2 n(z)\tilde{u} = -\delta(z)h(k), \tag{4}$$

where

$$\tilde{u} := \int_{-\infty}^{\infty}\!\!\int \exp[\mathrm{i}(\lambda_1 x_1 + \lambda_2 x_2)]u(x_1, x_2, z, k)\, \mathrm{d}x_1\, \mathrm{d}x_2,$$

$$\Lambda = (\lambda_1, \lambda_2) \qquad \lambda = (\lambda_1^2 + \lambda_2^2)^{1/2}.$$

Let us write (4) as

$$\tilde{u} = h(k)g + k^2 \int_{-\infty}^{\infty} g(z-z')n(z')\tilde{u}\,\mathrm{d}z' \qquad g := \frac{\exp(-\lambda|z|)}{2\lambda}. \tag{5}$$

Equation (5) is uniquely solvable by iterations if k is sufficiently small (see [1] for details and estimates). Therefore

$$\lim_{k\to 0}(\tilde{u} - h(k)g)k^{-2} = h(0)\int_{-\infty}^{\infty} \frac{\exp(-\lambda|z-z'|)}{2\lambda}\, n(z')\,\frac{\exp(-\lambda|z'|)}{2\lambda}\,\mathrm{d}z'. \tag{6}$$

Let us denote the left side of (6) at $z=0$ by $\varphi(\lambda)$. This is a known function since $\tilde{u}(\lambda_1, \lambda_2, 0, k)$ is known. Equation (6) at $z=0$ reads

$$\int_{-\infty}^{\infty} \exp(-2\lambda|z'|)n(z')\,\mathrm{d}z' = \psi(\lambda) := 4\lambda^2\varphi(\lambda)A \qquad A := h^{-1}(0). \tag{7}$$

Since $n(z)=1$ for $z>0$ by the assumption, one can write (7) as

$$\int_{-\infty}^{0} \exp(-2\lambda|z'|)n(z')\,\mathrm{d}z' = 4\lambda^2\varphi(\lambda)A - (2\lambda)^{-1} \qquad \lambda > 0. \tag{8}$$

Finally, set $2\lambda = p$, $z' = -\zeta$, $n(z') = m(\zeta)$, $4\lambda^2\varphi(\lambda := \psi(p)$ and write (8) as

$$\int_{0}^{\infty} \exp(-p\zeta)m(\zeta)\,\mathrm{d}\zeta = A\psi(p) - p^{-1} \qquad p > 0. \tag{9}$$

Equation (9) determines $m(\zeta)$ uniquely if A is known. From (5) it follows that

$$\lim_{k\to 0}\tilde{u}(\lambda_1, \lambda_2, 0, k) = h(0)/2\lambda \qquad \text{so that } A = [2\lambda \lim_{k\to 0}\tilde{u}(\lambda_1, \lambda_2, 0, k)]^{-1}. \tag{10}$$

Thus, given the data $u(x_1, x_2, 0, k)$ one finds $\tilde{u}(\lambda_1, \lambda_2, 0, k)$ (by Fourier transform), then A by formula (10), then $m(\zeta)$ by taking the inverse Laplace transform of the right side of (9), then $n(z)$ by the formula $n(z) = m(-z)$, $z<0$; $h(k)$ can be found for all k as follows. Set $z=0$ in (5) and write (5) as

$$h(k) = 2\lambda\tilde{u}(\lambda_1, \lambda_2, 0, k) - \int_{-\infty}^{\infty} \exp(-2\lambda|z'|)n(z')\tilde{u}(\lambda_1, \lambda_2, z', k)\,\mathrm{d}z'. \tag{11}$$

Let $\lambda \to +\infty$ in (11) then

$$h(k) = \lim_{\lambda\to +\infty} 2\lambda\tilde{u}(\lambda_1, \lambda_2, 0, k). \tag{12}$$

Formula (12) gives $h(k)$ directly from the data, so that $h(k)$ is found independently from $n(z)$. This $h(k)$ can be used on the left side of (6).

Let us summarise the results.

Theorem 1. If $n(z)=1$ for $z>0$ and conditions (2) and (3) hold, then the data $u(x_1, x_2, 0, k)$ determine $n(z)$ for $z<0$ and $h(k)$ uniquely. The analytical recovery of $n(z)$ and $h(k)$ is described above and is based on the formulae (9), (10) and (12).

A similar argument is applicable in the three-dimensional case. Namely, assume that

$$[\nabla^2 + k^2 + k^2 v(x)]u(x, y, k) = -\delta(x-y)h(k) \qquad \text{in } R^3 \tag{13}$$

where $v(x) \in L^2$ and $v(x)=0$ for $x_3 \geqslant 0$ or $|x| \leqslant r$, where $r>0$ is an arbitrary large number. Suppose that the data are the values $u(x_1, x_2, x_3=0, y_1, y_2, y_3=0, k)$ given for all $x^1, y^1 \in P$ and $k>0$. Here $x^1 = (x_1, x_2, 0)$, $P = \{x\colon x_3 = 0\}$.

The integral equation for u is

$$u(x, y, k) = h(k)G(x, y, k) + k^2 \int G(x, \xi, k)v(\xi)u(\xi, y, k)\,\mathrm{d}\xi \tag{14}$$

where $G := \exp(\mathrm{i}k|x-y|)/4\pi|x-y|$. One can solve equation (14) by iterations for sufficiently small k and prove that (see [2, p 218])

$$(4\pi)^2 \lim_{k\to 0} \frac{u-h(k)G}{k^2} = h(0) \int \frac{v(\xi)\,\mathrm{d}\xi}{|x-\xi||\xi-y|}. \tag{15}$$

Set $x=x^1, y=y^1$ in (15). Then the left side of (15), which we denote by $f(x^1, y^1)$, is known. If $h^{-1}(0) := A$ is known, then equation (15) for $x=x^1$ and $y=y^1$ becomes

$$\int \frac{v(\xi)\,\mathrm{d}\xi}{|x^1-\xi||y^1-\xi|} = Af(x^1, y^1) \qquad \forall\, x^1, y^1 \in P. \tag{16}$$

Equation (16) is solved analytically for $v(\xi)$ in [2, pp 220–2] where it is proved that $v(\xi)$ is determined uniquely by the data on the right side of (16). For A one derives from (14) the formula

$$A = \lim_{x^1\to y^1} [4\pi|x^1-y^1|u(x^1, y^1, 0)]^{-1}. \tag{17}$$

For $h(k)$ one derives from (14) the formula

$$h(k) = \lim_{x^1\to y^1} [4\pi|x^1-y^1|u(x^1, y^1, k)]. \tag{18}$$

The remark below formula (12) applies here as well. Let us summarise the result.

Theorem 2. The functions $v(x)$ and $h(k)$ are uniquely determined by the data $u(x^1, y^1, k)$ known for all $x^1, y^1 \in P$ and all $k > 0$. The analytical recovery of $v(x)$ and $h(k)$ is based on the formulae (16)–(18).

Remark. The assumption $h(0)\neq 0$ can be removed. If $h^{(j)}(0)=0$, $0\leqslant j\leqslant N-1$, $h^{(N)}(0)\neq 0$, then let $k^{-N}\tilde{u} := \tilde{w}$, $k^{-N}h(k)$: $h_N(k)$, $h_N(0)\neq 0$. The argument below formula (4) is applicable to $\tilde{w}(k)$. A similar remark holds in the 3D case. Finally, note that in practice $H(t)$ vanishes for large t. In this case $h(k)$ is analytic and cannot have zero of infinite order at $k=0$, so that there is an N such that $h^{(N)}(0)\neq 0$.

The author thanks ONR for support.

References

[1] Ramm A G 1987 An inverse problem for the Helmholtz equation in a semi-infinite medium *Inverse Problems* **3** L19–22

[2] Ramm A G 1986 *Scattering by Obstacles* (Dordrecht: Reidel)