Inverse Problems 3 (1987) L53–L57. Printed in the UK

LETTER TO THE EDITOR

Necessary and sufficient conditions for a function $A(\theta', \theta, k)$ to be the scattering amplitude corresponding to a reflecting obstacle

A G Ramm
Mathematics Department, Kansas State University, Manhattan, KS 66506, USA

Received 13 February 1987

Abstract. Necessary and sufficient conditions for a given function $A(\theta', \theta, k)$ to be the scattering amplitude corresponding to a reflecting obstacle are given.

Let $D \subset R^3$ be a finite domain (obstacle) with a smooth boundary Γ. Consider the scattering problem

$$(\nabla_x^2 + k^2)\psi(\theta, k, x) = 0 \quad \text{in} \quad \Omega = R^3 \backslash D \qquad k > 0 \tag{1}$$

$$\psi = 0 \quad \text{on} \quad \Gamma \qquad \psi = \psi_0 + v \qquad \psi_0 = \exp(\mathrm{i}k\theta \cdot x) \tag{2}$$

$$v = gA_\Gamma(\theta', \theta, k) + \mathrm{o}(r^{-1}) \qquad r = |x| \to \infty, \quad \theta' = xr^{-1}, \quad g = r^{-1}\exp(\mathrm{i}kr). \tag{3}$$

Here $\theta, \theta' \in S^2$, the unit sphere in R^3, the function $A_\Gamma(\theta', \theta, k)$ is called the scattering amplitude corresponding to the obstacle with boundary Γ.

It is well known [1] that

$$A_\Gamma = A(\theta', \theta, k) = -(4\pi)^{-1} \int_\Gamma \mathrm{d}s \exp(-\mathrm{i}k\theta' \cdot s)\partial\psi(\theta, k, s)/\partial N_s \tag{4}$$

where N_s is the outer unit normal. This function satisfies the following equations [1]:

$$\overline{A(\theta', \theta, -k)} = A(\theta', \theta, k) \quad k \text{ real} \qquad \text{(realness)} \tag{5}$$

$$A(\theta', \theta, k) = A(-\theta, -\theta', k) \qquad \text{(reciprocity)} \tag{6}$$

$$A(\theta', \theta, k) - A(\theta, \theta', -k) = \mathrm{i}k(2\pi)^{-1} \int_{S^2} A(\theta'', \theta, k)\, A(\theta'', \theta, -k)\mathrm{d}\theta'' \tag{7}$$

(optical theorem), which are necessary conditions for $A(\theta', \theta, k)$ to be a scattering amplitude corresponding to an obstacle. The bar in (5) denotes the complex conjugate.

The inverse problem consists of finding Γ from knowledge of $A(\theta', \theta, k)$. This problem is discussed in [1] where references are given. The basic uniqueness theorem can be found in [1].

Lemma 1. If $A = A_\Gamma$ is known for an infinite sequence θ_j, $\theta_j \neq \theta_{j'}$ for $j \neq j'$, which has a limit point, for all $\theta' \in \tilde{S}^2$, where $\tilde{S}^2$ is a solid angle, and for a fixed $k > 0$ then Γ is determined uniquely.

Formulae for practical recovery of Γ from $A(\theta', \theta, k)$ known at one large k (large in the sense $ka \gg 1$, where $a = \text{diam } D$) are given in [1].

Conditions for the set $\{A(\theta', \theta, k)\}$ to be dense in $L^2(S^2)$ when $k>0$ is fixed and θ runs through S^2 (or the boundary data run through a dense set in $L^2(\Gamma)$) are given in [2].

In this Letter we give a characterisation of the class of scattering amplitudes, i.e. we give necessary and sufficient conditions for a given function $A(\theta', \theta, k)$, $\theta', \theta \in S^2$, $k>0$ and fixed, to be the scattering amplitude corresponding to a reflecting obstacle.

Our characterisation is obtained by the method first given in [3]. The starting point is the equation in $L^2(S^2)$:

$$\psi_+ = S\psi_- \qquad S = I + ik(2\pi)^{-1}A \qquad A\psi := \int_{S^2} A(\theta', \theta, k)\psi(\theta')\mathrm{d}\theta' \quad (8)$$

where $\psi_+ = \psi$ and $\psi_- = \psi(-\theta, -k, x)$. Let us write (8) as

$$v_+(\theta, k, x) = v_+(-\theta, -k, x) + ik(2\pi)^{-1}Av_+(-\theta, -k, x) + v_0 \qquad \forall x \in R^3 \quad (9)$$

$$v_0 := ik(2\pi)^{-1}A\psi_0(\theta, k, x).$$

Here

$$v_+(\theta, k, x) := \psi_+(\theta, k, x) - \exp(ik\theta\cdot x).$$

Note that if $A(\theta', \theta, k) = A_\Gamma(\theta', \theta, k)$, that is, if A is the scattering amplitude corresponding to Γ, then

equation (9) has a solution $v_+(\theta, k, x)$ such that the function

$$\psi := \psi_0 + v_+ \quad \text{satisfies equations (1)–(3).} \qquad (10)$$

In particular, Γ can be defined as the closed surface of zeros of the function ψ constructed in (10), and (10) is a necessary condition for $A(\theta', \theta, k)$ to be the scattering amplitude corresponding to a reflecting obstacle with the boundary Γ. We will write $A \in \mathscr{S}$ if A is a scattering amplitude corresponding to a reflecting obstacle. The result is as follows.

Theorem 1. $A \in \mathscr{S}$ iff (10) holds. Equation (9) has at most one solution with the properties (10), and the function A_Γ in condition (3) for the solution of (9) is equal to the kernel $A(\theta', \theta, k)$ in equation (9).

Remark 1. Theorem 1 gives a characterisation of the class of scattering amplitudes and a theoretical method to solve the inverse problem. This method consists of finding all the solutions to (9) and checking for each of them whether conditions (1)–(3) hold. If there is no solution for which (1)–(3) hold then $A(\theta', \theta, k) \notin \mathscr{S}$. If there is a solution to (9) for which (1)–(3) hold then this solution is unique, $A \in \mathscr{S}$, moreover $A = A_\Gamma$, where A_Γ is the coefficient in condition (3), and Γ can be found as (the unique) closed surface of zeros of $\psi = \psi_0 + v_-$, where v_+ is the solution to (9) with properties (10). Uniqueness of Γ follows from lemma 1.

Proof of theorem 1. Necessity of (10): $A \in \mathscr{S} \Rightarrow (10)$. This implication is known and was discussed above.
Sufficiency of (10): $(10) \Rightarrow A \in \mathscr{S}$. Assume that (9) has a solution with properties (10).
Claim 1. This solution is unique. Suppose there are two (or more) solutons v_j, $j = 1, 2$, with properties (10). Then $w := v_1 - v_2$ solves the homogeneous equation

$$w(\theta, k, x) = w(-\theta, -k, x) + ik(2\pi)^{-1}Aw(-\theta, -k, x) \qquad \forall x \in R^3. \quad (9')$$

Using condition (3) one obtains from (9′) that

$$g(A_1 - A_2) = \bar{g}B + o(r^{-1}) \qquad \text{as } r = |x| \to \infty \quad (11)$$

where the bar denotes the complex conjugate and the value of B is not important, and where $A_j = A_{\Gamma_j}$ is the coefficient in (3) corresponding to v_j. It follows from (11) that $A_1 = A_2$. This and lemma 1 imply that $\Gamma_1 = \Gamma_2$. Thus $v_1 = v_2$ as claimed. Let us denote $v := v_1 = v_2$, $\Gamma := \Gamma_1 = \Gamma_2$ and $A_\Gamma := A_1 = A_2$.
Claim 2. $A_\Gamma = A$, where A is the kernel in equation (9). Indeed, the solution v to equation (9) with properties (1)–(3) also satisfies equation (9) with A_Γ in place of A. These two equations can be written as

$$\psi_+ = S\psi_- = S_\Gamma \psi_- \qquad \text{where} \qquad S_\Gamma := I + \mathrm{i}k(2\pi)^{-1}A_\Gamma. \tag{12}$$

Thus $(S - S_\Gamma)\psi_- = 0$, or

$$\int_{S_2} \mathrm{d}\theta' [A(\theta', \theta, k) - A_\Gamma(\theta', \theta, k)]\psi_-(\theta', k, x) = 0 \qquad \forall\, x \in R^3. \tag{13}$$

The desired conclusion, $A = A_\Gamma$, follows from lemma 2, which is formulated and proved below. Theorem 1 is proved.

Lemma 2. Let $f \in L^2(S^2)$ and

$$\int_{S^2} \mathrm{d}\theta\, f(\theta)\psi_-(\theta, k, x) = 0 \qquad \forall\, x \in \Omega_R := \{x : |x| \geqslant R\} \tag{14}$$

where $R > 0$ is an arbitrary large fixed number. Then $f = 0$.

Proof. If (14) holds with ψ_0 in place of ψ_- then one concludes that $f = 0$. Indeed, let $u_0(x) := \int_{S^2} f(\theta)\psi_0(\theta, k, x)\mathrm{d}\theta$. Then u_0 solves equation (1) in R^3 and $u_0 = 0$ in Ω_R. By the unique continuation theorem for elliptic equations $u_0 = 0$ in R^3. By the uniqueness theorem for the Fourier transform of distributions one concludes that $f(\theta) = 0$.

If (14) holds then denote by $u := u(x, k)$ the integral in (14), note that $(\nabla^2 + k^2)u = 0$ in Ω and $u = 0$ in Ω_R. By the unique continuation property for elliptic equations, $u = 0$ in Ω. Multiply (14) by an arbitrary $h \in L^2(\Omega)$ and integrate over Ω to get

$$0 = \int_{S^2} \mathrm{d}\theta\, f(\theta)H(\theta, k) \qquad H := \int_\Omega \mathrm{d}x\, \psi_-(\theta, k, x)h(x). \tag{15}$$

The mapping $h \mapsto H$ is a bijection of $L^2(\Omega)$ onto $L^2(R^3)$ [1]. Therefore, $H(\theta, k)$ runs through all of $L^2(R^3)$ when h runs through all of $\mathrm{L}^2(\Omega)$.

If $k > 0$ is fixed and $H(\theta, k)$ runs the set of smooth rapidly decaying functions then the set of $H(\theta, k)$, $k = \text{constant} > 0$, is dense in $L^2(S^2)$ and (15) implies that $f = 0$. For example, one can take $H(\theta, k) = p(\theta)(1 + k^2)^{-n}$, $n > 2$, and $p(\theta) \in L^2(S^2)$ is arbitrary. These $H(\theta, k) \in L^2(R^3)$ are admissible. Lemma 2 is proved.

We will now give another characterisation of the scattering data. Let us derive an analogue of the Marchenko equation in the 3D inverse potential scattering. Let $\gamma := \psi\psi_0^{-1}$. Equation (8) can be written as

$$\begin{aligned}
\gamma_+(\theta, k, x) &= \gamma_+(-\theta, -k, x) \\
&\quad + \mathrm{i}k(2\pi)^{-1} \int_{S^2} A(\theta', \theta, k) \exp[\mathrm{i}k(\theta' - \theta) \cdot x][\gamma_+(-\theta', -k, x) - 1]\, \mathrm{d}\theta' \\
&\quad + \mathrm{i}k(2\pi)^{-1} \int_{S^2} A(\theta', \theta, k) \exp[\mathrm{i}k(\theta' - \theta) \cdot x]\mathrm{d}\theta'.
\end{aligned} \tag{16}$$

Denote

$$\eta(\theta, \alpha, x) := (2\pi)^{-1} \int_{-\infty}^{\infty} [\gamma_+(\theta, k, x) - 1] \exp(-ik\alpha)\, dk \tag{17}$$

$$M(\theta', \theta, \alpha, x) := (2\pi)^{-2} i \int_{-\infty}^{\infty} k \exp(-ik\alpha) \left(\int_{S^2} A(\theta', \theta, k) \exp[ik(\theta' - \theta) \cdot x]\, d\theta' \right) dk \tag{18}$$

$$\eta_0(\theta, \alpha, x) := (2\pi)^{-2} \int_{-\infty}^{\infty} \exp(-ik\alpha) \left(ik \int_{S^2} A(\theta', \theta, k) \exp[ik(\theta' - \theta) \cdot x]\, d\theta' \right) dk. \tag{19}$$

Fourier transform (16) to get

$$\eta(\theta, \alpha, x) = \eta(-\theta, -\alpha, x) + M\eta + \eta_0(\theta, \alpha, x) \qquad -\infty < \alpha < \infty \tag{20}$$

where

$$M\eta := \int_{-\infty}^{\infty} \int_{S^2} M(-\theta', \theta, \alpha + \alpha') \eta(\theta', \alpha', x)\, d\theta'\, d\alpha'. \tag{21}$$

For $\alpha > 0$ one has $\eta(-\theta, -\alpha, x) = 0$ because $\gamma(\theta, k, x) - 1$ is analytic in $\mathrm{Im}\, k > 0$. Thus (20) implies

$$\eta = M\eta + \eta_o \qquad \alpha > 0. \tag{22}$$

This equation is analogous to the Marchenko equation derived in [4].

One can give another characterisation of the class of scattering amplitudes using equation (22).

Theorem 2. $A \in \mathscr{S}$ iff the following conditions (C) hold:

(i_1) there exists a solution to (20) such that $\eta = 0$ for $\alpha < 0$.

(i_2) the function $\psi := \psi_0[1 + \int_0^{\infty} \exp(ik\alpha)\eta(\theta, \alpha, x)\, d\alpha]$ satisfies conditions (1)–(3). (C)

There is exactly one solution to equation (20) with the properties (i_1) and (i_2). Iff $A \in \mathscr{S}$ equation (22) has a solution η such that if η is defined for $\alpha < 0$ by the equation $\eta = 0$ for $\alpha < 0$ then this η solves equation (20) and satisfies conditions (C).

Proof. Necessity: $A \in \mathscr{S} \Rightarrow$(C). This part has been derived above.

Sufficiency: (C)$\Rightarrow A \in \mathscr{S}$.

Claim 1. If (C) hold then (20) has exactly one solution with properties (i_1), (i_2). Indeed, one solution η_1 exists by the assumption (i_1). Suppose there is another one, η_2. The difference of the solutions, $\eta_1 - \eta_2 := \eta$ satisfies the homogeneous equation (20), and its inverse Fourier transform, γ, satisfies the equation

$$\psi_0 \gamma(\theta, k, x) = \psi_0 \gamma(-\theta, -k, x) + ik(2\pi)^{-1} \int_{S^2} A(\theta', \theta, k) v(-\theta', -k, x)\, d\theta' \tag{23}$$

or

$$v(\theta, k, x) = v(-\theta, -k, x) + ik(2\pi)^{-1} \int_{S^2} A(\theta', \theta, k) v(-\theta', -k, x)\, d\theta' \qquad \forall x \in R^3 \tag{24}$$

where $v = \psi - \psi_0$ and $\psi = \psi_0 \gamma$. When $|x| \to \infty$ equation (24) takes the form (11) and yields the same conclusions: $A_1 = A_2 := A_\Gamma$, $v_1 = v_2$ and $\Gamma_1 = \Gamma_2 := \Gamma$.

Claim 2. $A_\Gamma = A$, where $A = A(\theta', \theta, k)$ is the kernel in equation (20). The proof of this claim is analogous to the proof in theorem 1 and is therefore omitted. Theorem 2 is proved.

Example. Let us find whether the function $A(\theta', \theta, k) := a(k) \in \mathscr{S}$. Here $a(-k) = a(k)$ and $\mathrm{Im}\, a = k|a(k)|^2$ for k real so that conditions (5)–(7) are satisfied. For example, $a(k) = \mathrm{i}(k+\mathrm{i})^{-1}$. Equation (9) for this A takes the form

$$v(\theta, k, x) = v(-\theta, -k, x) + \mathrm{i}k(2\pi)^{-1} a(k)\,(\alpha - \mathrm{i}\beta) + \mathrm{i}k(2\pi)^{-1} a(k) 4\pi j_0(kr) \tag{25}$$

where

$$\alpha - \mathrm{i}\beta := \int_{S^2} v(-\theta, -k, x)\,\mathrm{d}\theta \qquad \alpha + \mathrm{i}\beta = \int_{S^2} v(\theta, k, x)\,\mathrm{d}\theta.$$

The reader can check that equation (25) does not have solutions which satisfy all the conditions (1)–(3). Therefore by theorem 1 the function $A(\theta', \theta, k) = a(k)$ is not the scattering amplitude corresponding to an obstacle.

Note that the necessary conditions (5)–(7) for $A \in \mathscr{S}$ are satisfied by $A = \mathrm{i}(k+\mathrm{i})^{-1}$, and this A can be continued meromorphically on the complex plane k and is analytic in the upper half-plane, as scattering amplitudes $A \in \mathscr{S}$ should be. Therefore, it is not obvious that $a(k) \notin \mathscr{S}$ if one does not use theorem 1 or some additional considerations.

The author thanks ONR for support and SFB 123 for hospitality.

References

[1] Ramm A G 1986 *Scattering by Obstacles* (Dordrecht: Reidel)
[2] Ramm A G 1984 On a property of the set of radiation patterns *J. Math. Anal. Appl.* **98** 92–8
[3] Ramm A G 1986 Characterisation of the scattering data in the three-dimensional inverse scattering problems *Preprint*
[4] Newton R G 1981 Inverse scattering in three dimensions *J. Math. Phys.* **22** 2191–200