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LETTER TO THE EDITOR

A characterisation of the scattering data in the 3D inverse scattering problem

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Abstract. A necessary and sufficient condition is given for the given function $A(\theta', \theta, k)$ to be the scattering amplitude corresponding to a potential from a class Q and a condition is given for Newton's equation to have exactly one solution with the desired properties.

Newton [1] developed a theory of inverse scattering by a potential in three dimensions (3D). His work is a significant contribution to the field. There are still some open questions in the theory. In particular, the characterisation of the scattering data, that is, necessary and sufficient conditions on the given function $A(\theta', \theta, k)$, $\theta', \theta \in S^2$ (the unit sphere in R^3), $k \in R_+ = [0, \infty)$, are not found. A partial characterisation given in [1b, p 2194] includes a certain compatibility condition on the solutions to Newton's equation (equation (4.4) in [1a]), the miracle condition (4.3) in [1a]. It also includes the assumption that (N) (= the Newton equation) has at most one solution. No conditions on $A(\theta', \theta, k)$ (or $q(x)$) are known to guarantee that this assumption holds. The role of analyticity in the inversion procedure is clear in the derivation of equation (N). The analyticity we refer to is analyticity (or, in the presence of bound states, meromorphicity) of the scattering solution to the Schrödinger equation

$$\nabla^2\psi + k^2\psi - q(x)\psi = 0 \qquad \psi = \psi_0 + v \qquad \psi_0 = \exp(\mathrm{i}k\theta \cdot x) \tag{1}$$

$$v = gA_q(\theta', \theta, k) + \mathrm{o}(r^{-1}) \quad \text{as } |x| = r \to \infty, \quad xr^{-1} = \theta', \quad k > 0, \quad g := r^{-1}\exp(\mathrm{i}kr) \tag{2}$$

in the half-plane $\mathrm{Im}\, k > 0$. The purpose of this Letter is to give a complete characterisation of the scattering data.

We first recall the Newton equation and sketch its derivation for the convenience of the reader. The starting point is the well known relation

$$\psi(\theta, k, x) = \int_{S^2} S(\theta', \theta, k)\psi(-\theta', -k, x)\,\mathrm{d}\theta' \tag{*}$$

where

$$S(\theta', \theta, k) = \delta(\theta' - \theta) + \frac{\mathrm{i}k}{2\pi}A(\theta', \theta, k)$$

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is the kernel of the S matrix. Define

$$\gamma := \psi \exp(-ik\theta \cdot x)) = \gamma(\theta, k, x)$$

$$\widetilde{\gamma - 1} := \eta(\theta, \alpha, x) := (2\pi)^{-1} \int_{-\infty}^{\infty} (\gamma - 1) \exp(-ik\alpha)\, \mathrm{d}k$$

and

$$\eta_0 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{ik}{2\pi} \int_{S^2} A(\theta', \theta, k) \exp[ik(\theta' - \theta) \cdot x]\, \mathrm{d}\theta' \right) \exp(-ik\alpha)\, \mathrm{d}k.$$

Multiply $(*)$ by ψ_0^{-1}, subtract 1 from both sides of the equation, and take the Fourier transform of the resulting equation to get

$$\eta(\theta, \alpha, x) = \eta(-\theta, -\alpha, x) + \int_{-\infty}^{\infty} \int_{S^2} M(\theta', \theta, \alpha - \alpha', x)\, \eta(-\theta', -\alpha', x)\, \mathrm{d}\theta'\, \mathrm{d}\alpha' + \eta_0(\theta, \alpha, x) \tag{3}$$

$$M(\theta', \theta, \alpha, x) := \frac{i}{(2\pi)^2} \int_{-\infty}^{\infty} kA(\theta', \theta, k)) \exp[ik(\theta' - \theta) \cdot x] \exp(-ik\alpha)\, \mathrm{d}k. \tag{4}$$

If one changes in the integral in (3) $\theta' \to -\theta'$, $-\alpha' \to \beta$ and, most importantly, takes into account analyticity which in the absence of bound states says that $\gamma - 1$ is analytic in $\operatorname{Im} k > 0$ so that $\eta(-\theta, -\alpha, x) = 0$ for $\alpha > 0$, then one obtains Newton's equation (equation (2.1) in [1b]):

$$\eta = M\eta + \eta_0 \qquad M\eta := \int_0^{\infty} \int_{S^2} M(-\theta', \theta, \alpha + \beta, x)\, \eta(\theta', \beta, x)\, \mathrm{d}\theta'\, \mathrm{d}\beta \qquad \alpha > 0. \tag{N}$$

In order to avoid the discussion of the high-energy behaviour ($k \to \infty$) of $A(\theta', \theta, k)$ one can understand the Fourier transform (4) in the sense of distributions.

Let Q denote the class of potentials $q(x)$ such that $q(x) = \overline{q(x)}$ (the bar stands for the complex conjugate), $|q(x)| + |\nabla q(x)| \leqslant c(1 + |x|)^{-a}$, $a > 3$, and let Q_n denote the subset of potentials $q \in Q$ which produce n bound states.

Let us denote by S_n the assumption that $A(\theta', \theta, k)$ corresponds to a potential $q \in Q_n$, and write $A \in \mathscr{S}_n$ in this case. We assume $n = 0$ and denote by (C) the following set of conditions:

(i_1) there exists a solution η to (3) for $-\infty < \alpha < \infty$ such that $\eta = 0$ for $\alpha < 0$. (Note that this η solves (N) for $\alpha > 0$.)

(i_2) equation (1) is satisfied by $\psi := \psi_0(1 + \int_0^{\infty} \eta \exp(ik\alpha)\, \mathrm{d}\alpha)$; this defines $q(x)$: $q = \psi^{-1}(\nabla^2 + k^2)\psi$, and this ψ satisfies (2). (Note that (i_2) implies analyticity of ψ in $\operatorname{Im} k > 0$.)

Our basic result is as follows.

Theorem 1. $\mathscr{S}_0$ is equivalent to (C) and there is exactly one solution to (N) which satisfies conditions (C) if $\mathscr{S}_0$ holds.

In other words, if $A \in \mathscr{S}_0$ then conditions (C) hold and $A = A_q$, where A_q is defined in (2); conversely, if (C) hold for a given A, then $A \in \mathscr{S}_0$ and $A = A_q$, where A_q is defined in (2). Note that the following lemma is well known.

Lemma 1. If $A(\theta', \theta, k) \in \mathscr{S}_n$ is known for all $\theta', \theta \in S^2$ and all $k > 0$ then $q(x)$ is defined uniquely.

Proof of theorem 1. The part $\mathscr{S}_0 \Rightarrow$(C) follows from the theory of the direct scattering problem. Namely, if $q \in Q_0$ then conditions (C) are known to be satisfied (see e.g. [1] where conditions (C) are established; the assumptions on $q(x)$ are slightly stronger in [1] than here, but these stronger assumptions were needed in order to establish some additional properties of the solutions to (1); for our purposes the assumption $q \in Q_0$ is sufficient.)

Suppose now that A is such that (C) hold. We need to prove that then $A \in \mathscr{S}_0$, that (N) has exactly one solution with properties (C) and that $A = A_q$.

Suppose that there are two (or more) solutions to (N) with properties (C), and that $q_j(x)$, $A_j := A_{q_j}$, $j = 1, 2$, are the corresponding potentials and the scattering amplitudes defined in (2). Then $w := \eta_1 - \eta_2$ solves equation (N_0):$w = Mw$, and $w(\alpha) = 0$ for $\alpha < 0$. If one reverses the argument above then one starts with the equation

$$w(\theta, \alpha, x) = w(-\theta, -\alpha, x) + \int_{-\infty}^{\infty} \int_{S^2} M(\theta', \theta, \alpha - \alpha', x) w(-\theta', -\alpha', x)\, \mathrm{d}\theta'\, \mathrm{d}\alpha, \tag{5}$$

takes the inverse Fourier transform of (5) and obtains

$$v(\theta, k, x) = v(-\theta, -k, x) + \frac{\mathrm{i}k}{2\pi} \int_{S^2} A(\theta', \theta, k) v(-\theta', -k, x) \mathrm{d}\theta' \qquad \forall\, x \in R^3 \tag{6}$$

where $v(\theta, k, x) := v_1 - v_2$, and $v_j := \psi_j - \psi_0$, $j = 1, 2$. But (6) cannot hold $\forall\, x \in R^3$ unless $A_1 = A_2$. Indeed, the left-hand side of (6) has asymptotes $g(A_1 - A_2)$ as $|x| = r \to \infty$ (see (2)), while the right-hand side has asymptotes $\bar{g}B$, where the exact value of $B(\theta', \theta, k)$ is not important for the argument. Therefore $A_1 = A_2$ and, by Lemma 1, $q_1 = q_2$. So we have proved that (N) has at most one solution with properties (C). Let us prove that $A = A_q$. The solution η with properties (C) generates the solution ψ to (1), (2), the potential $q(x)$ and the function A_q, and it is known from the theory of the direct scattering problem that ψ solves equation $(*)$ with $S_q = I + (\mathrm{i}k/2\pi)A_q$ in place of S. Since ψ also solves equation $(*)$, one concludes that

$$0 = \int_{S^2} [A(\theta', \theta, k) - A_q(\theta', \theta, k)] \psi(-\theta', -k, x)\, \mathrm{d}\theta' \qquad \forall\, x \in R^3. \tag{7}$$

But (7) implies that $A(\theta', \theta, k) = A_q(\theta', \theta, k)$ according to lemma 2 in [2].

Lemma 2. If

$$\int_{S^2} f(\theta') \psi(-\theta', -k, x)\, \mathrm{d}\theta' = 0 \qquad \forall\, x \in R^3$$

and $f \in L^2(S^2)$ then $f = 0$.

Theorem 1 is proved. Our ideas are very similar to the ideas first presented in [2].

Remark 1. Conditions (C) are compatibility conditions on the scattering data, which are analogous to Newton's miracle. Their merit is that they allow one to give a characterisation of the scattering data.

Remark 2. Our method is valid in dimension $d \geqslant 2$ and can be generalised to the case $q \in Q_n$, $n > 0$.

Remark 3. In [3] a characterisation of the scattering data in the problem of scattering by an obstacle is given by the method presented in [2].

Remark 4. The reader may think that our basic compatibility conditions (C) are very strong, and so one could look for weaker compatibility conditions. But in fact these conditions are very natural and there are no weaker compatibility conditions because, as we have proved, our conditions are necessary and sufficient for the given function $A(\theta', \theta, k)$ to be the scattering amplitude for an underlying potential $q \in Q_0$. For example, the compatibility condition used in [1a], the miracle of Newton, is not alone sufficient for characterisation of the scattering data. Although one may hope to find a more constructive compatibility condition, any compatibility condition which gives a characterisation of the scattering data has to be equivalent to our compatibility conditions.

Remark 5. In the case $n>0$ conditions (C_n) are: (i_{1n}) equation (3) has a solution η; (i_{2n}) equation (1) is satisfied by $\psi_i = \psi_0 \; (1 + \int_{-\infty}^{\infty} \eta \exp(\mathrm{i}\kappa\alpha) \, \mathrm{d}\alpha)$, this defines $q := \psi^{-1}(\nabla^2 + k^2)\psi$; ψ satisfies (2) and $q \in Q_n$. Theorem 1 in this case reads: $\mathscr{S}_n$ is equivalent to (C_n). Equation (N) in this case is: $\eta = M\eta + \eta_0 + \Sigma_{j=1}^{n} s_j(-\theta, x) \exp(-\alpha\kappa_j)$, $\alpha > 0$, where $\mathrm{i}\kappa_j$, $\kappa_j > 0$, are (simple) poles of $\psi(-\theta, k, x)$ on the complex plane k, and $s_j(-\theta, x)$ are the residues of $\gamma(-\theta, k, x) - 1$ at $k = \mathrm{i}\kappa_j$. Given $A(\theta', \theta, k)$, $\theta', \theta \in S^2$, $0 < k < \infty$, one can find $\mathrm{i}\kappa_j$ as poles of $A(\theta, \theta, k)$ but practically it is difficult to find $s_j(-\theta, x)$ (cf [1a]).

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