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LETTER TO THE EDITOR

Characterisation of the low-frequency scattering data in the inverse problem of geophysics

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Abstract. Let $(\nabla^2+k^2+k^2v(x))u(x,y,k)=-\delta(x-y)$, $x\in R^3$. Here u is the acoustic field generated by the point source located at the point y in an inhomogeneous medium with refraction coefficient $1+v(x)$. Assume that (*) $\{v(x)\in L^2(R^3)$, $v(x)=0$ if $x_3\geqslant 0$ or $|x_j|>R$, $1\leqslant j\leqslant 3$, where $R>0$ is an arbitrary large number$\}$. The low-frequency scattering data are defined to be the function $f(x^1,y^1):=16\pi^2\lim_{k\to 0}k^{-2}u_s(x^1,y^1,k)$ for all $x^1, y^1\in P=\{x: x_3=0\}$ and $0<k<k_0$, where $k_0>0$ is a fixed number, and $u_s(x^1,y^1,k)$ is the scattered field. Necessary and sufficient conditions are given for a function $f(x^1,y^1)$ to be the low-frequency scattering data corresponding to $v(x)$ which satisfies conditions (*). Here $x^1=(x_1,x_2,0)$.

Let

$$(\nabla^2+k^2+k^2v(x))u(x,y,k)=-\delta(x-y) \qquad x\in R^3, \quad k>0. \tag{1}$$

Assume that

$$v(x)=0 \quad \text{if } x_3\geqslant 0 \text{ or } |x_j|>R \qquad j=1,2,3, \quad v(x)\in L^2(R^3). \tag{2}$$

The scattered field measured on the plane $P=\{x: x_3=0\}$ is

$$u_s(x^1,y^1,k):=u(x^1,y^1,k)-u_0(x^1,y^1,k) \qquad u_0(x,y,k):=g:=\frac{\exp(ik|x-y|)}{4\pi|x-y|} \tag{3}$$

where u_0 is the incident field, the field in the absence of the inhomogeneity $v(x)$. The inverse problem of geophysics consists in finding $v(x)$, the inhomogeneity, from the scattering data $u_s(x^1,y^1,k)$, $x^1, y^1\in P$, $0<k\leqslant k_0$. Here x^1 and y^1 run through all of P and $k_0>0$ is an arbitrary (small) fixed number.

This inverse problem was solved exactly (in closed form) by the author in 1983. The solution is presented in detail in [1, ch. 6], where the low-frequency scattering data are defined to be $f(x^1,y^1):=16\pi^2\lim_{k\to 0}k^{-2}u_s(x^1,y^1,k)$. The purpose of this Letter is to formulate necessary and sufficient conditions on the given function $f(x^1,y^1)$, $x^1,y^1\in P$, for this function to be the low-frequency scattering data corresponding to the inhomogeneity $v(x)$ which satisfies assumptions (2). In other words, the purpose is to give a characterisation of the low-frequency scattering data for the inverse problem of geophysics.

In order to do this, let us recall (without proofs) the solution given in [1, pp 218–22]. One writes (1) as

$$u=g+k^2\int g(x,z,k)v(z)u(z,y,k)\,dz \qquad \int=\int_{R^3}. \tag{4}$$

For sufficiently small k equation (4) is uniquely solvable by iterations and one proves that

$$f(x^1, y^1) := 16\pi^2 \lim_{k\to 0} k^{-2} u_s(x^1, y^1, k) = \int (|x^1 - z|\,|y^1 - z|)^{-1} v(z)\,\mathrm{d}z. \quad (5)$$

This integral equation for $v(z)$ is solved analytically in [1] (in closed form). The solution is given by the formula

$$v(z_1, z_2, z_3) = \tfrac{1}{4} \iint_{-\infty}^{\infty} \mathrm{d}p_1\,\mathrm{d}p_2\, \exp(-\mathrm{i}p\cdot z^1)(2\pi\mathrm{i})^{-1} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}q\, \exp(-qz_3)\phi(p_1, p_2, q) \quad (6)$$

where $c > 0$ is an arbitrary constant, $p\cdot z^1 = p_1 z_1 + p_2 z_2$,

$$\phi(p_1, p_2, q) = q^2 F(p_1, p_2, q/2, q/2) \quad (7)$$

$$F(p_1, p_2, p_3, p_4) = (2\pi)^{-4} \iint_{-\infty}^{\infty} f(x^1, y^1) \exp[\mathrm{i}(\lambda\cdot x^1 + \mu\cdot y^1)]\,\mathrm{d}x^1\,\mathrm{d}y^1 := \tilde{f}(\lambda, \mu) \quad (8)$$

$\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, $\mathrm{d}x^1 = \mathrm{d}x_1\,\mathrm{d}x_2$, $\mathrm{d}y^1 = \mathrm{d}y_1\,\mathrm{d}y_2$, and the function on the right-hand side of (8) is expressed in terms of the variables p_1, p_2, p_3, p_4 via the formulae

$$p_1 = \lambda_1 + \mu_1 \qquad p_2 = \lambda_2 + \mu_2$$
$$p_3 = (\lambda_1^2 + \lambda_2^2)^{1/2} = |\lambda| \qquad p_4 = (\mu_1^2 + \mu_2^2)^{1/2} = |\mu|. \quad (9)$$

The Jacobian J of the mapping $(\lambda_1, \lambda_2, \mu_1, \mu_2) \to (p_1, p_2, p_3, p_4)$ is $J = (|\lambda|\,|\mu|)^{-1}(\mu_1\lambda_2 - \mu_2\lambda_1)$. Note that $J \neq 0$ if the vectors μ and λ are linearly independent. Formulae (6)–(9) are derived in [1, pp 221–2].

From formulae (6), (7) and (9) one obtains the following characterisation of the low-frequency data $f(x^1, y^1)$ in the inverse problem of geophysics.

Theorem. For a function $f(x^1, y^1)$, $x^1, y^1 \in P$, to be the low-frequency data corresponding to an inhomogeneity $v(z)$ which satisfies conditions (2) it is necessary and sufficient that the function $\phi(p_1, p_2, q)$ defined by formulae (7)–(9) be an entire function of its three variables of exponential type $\leqslant R$ such that

$$\sup_{\sigma\geqslant 0} \iiint_{-\infty}^{\infty} |\phi(p_1, p_2, \sigma + \mathrm{i}\tau)|^2\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}\tau < \infty. \quad (10)$$

Proof. The following well-known Paley–Wiener theorems [2] are needed for the proof.

1. A function $\psi(p)$ is an entire function of exponential type $\leqslant R$ such that

$$\int_{-\infty}^{\infty} |\psi(p)|^2\,\mathrm{d}p < \infty \qquad \text{iff} \quad \psi(p) = \int_{-R}^{R} \exp(\mathrm{i}p\cdot x) f(x)\,\mathrm{d}x$$

where $f(x) \in L^2(-R, R)$.

2. A function $\psi(p)$ is an entire function of exponential type $\leqslant R$ such that

$$\sup_{\sigma\geqslant 0} \int_{-\infty}^{\infty} |\psi(\sigma + \mathrm{i}\tau)|^2\,\mathrm{d}\tau < \infty \qquad \text{iff} \quad \psi(p) = \int_0^R \exp(-pt) f(t)\,\mathrm{d}t$$

where $f \in L^2(0, R)$. Formula (6) can be written as

$$\varphi(p_1, p_2, q) = \pi^{-2} \int_0^{\infty} \left(\iint_{-\infty}^{\infty} v(z_1, z_2, -z_3) \exp(\mathrm{i}p\cdot z^1)\,\mathrm{d}z^1 \right) \exp(-qz_3)\,\mathrm{d}z_3. \quad (11)$$

From the theorems 1 and 2 it follows that conditions (2) are satisfied iff $\varphi(p_1, p_2, q)$ is an entire function of its three variables of exponential type $\leqslant R$ such that inequality (10) holds.

Remark. Under the assumptions of the theorem above the function $\varphi(p_1, p_2, q)$ is bounded in the half-plane Re $q \geqslant 0$. Indeed, it follows from (11) that

$$\varphi(p_1, p_2, q) = \int_0^R d\eta \exp(-q\eta) w(p_1, p_2, \eta)$$

$$w := \pi^{-2} \int_{-R}^{R} \int_{-R}^{R} v(z_1, z_2, -\eta) \exp(\mathrm{i} p \cdot z^1) \mathrm{d}z_1 \, \mathrm{d}z_2.$$

Thus, Cauchy's inequality yields

$$\int_0^R \mathrm{d}\eta |w|^2 \leqslant c_1(R) \int_0^R \int_{-R}^{R} \int_{-R}^{R} |v(z_1, z_2, -\eta)|^2 \, \mathrm{d}z_1 \, \mathrm{d}z_2 \mathrm{d}\eta \leqslant C(R) \qquad C(R) = \text{constant} > 0$$

since by assumption $v \in L^2$.

References

[1] Ramm A G 1986 *Scattering by Obstacles* (Dordrecht: Reidel)

[2] Fuks B A 1963 *Theory of Analytic Functions of Several Complex Variables* (Providence, RI: Am. Math. Soc.)