

LETTER TO THE EDITOR

A method for solving inverse diffraction problems

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Abstract. A method is given for finding the surface of an obstacle from knowledge of the scattering amplitude at a fixed frequency, fixed direction of the incident plane wave, and all directions of the scattered wave. The surface is assumed to be smooth, star-shaped and connected.

Consider the problem

$$(\nabla^2 + k^2)u = 0 \quad \text{in } \Omega = R^3 \setminus D, \quad k > 0 \tag{1}$$

$$u = 0 \quad \text{on } \Gamma = \partial D \tag{2}$$

$$u = \exp(ikn \cdot x) + v \tag{3}$$

$$v \sim r^{-1} \exp(ikr) f(n, \nu, k) \quad \text{as } |x| = r \rightarrow \infty, \quad r^{-1}x = \nu. \tag{4}$$

Here $n, \nu \in S^2$, S^2 is the unit sphere in R^3 , D is a bounded domain with a smooth connected boundary Γ . It is well known that problem (1)–(4) is uniquely solvable, see e.g. [1]. The function $f(n, \nu, k)$ is called the scattering amplitude. The problem we discuss in this paper is that of recovery of Γ from knowledge of $f(\nu) = f(n, \nu, k)$ for a fixed n and k , and ν running through S^2 .

Some related inverse problems are discussed in [1, ch. 2], where further references are given.

To solve the above problem we assume that Γ is star-shaped, i.e. the equation of Γ can be written as $r = r(\nu)$ in a coordinate system with the origin in D . We write

$$f(\nu) = \sum_{p=0}^{\infty} f_p Y_p(\nu) \tag{5}$$

where $Y_p(\nu)$ are the spherical harmonics normalised in $L^2(S^2)$. Write

$$v = \sum_{p=0}^{\infty} f_p Y_p(\nu) h_p(kr) \quad r \geq R. \tag{6}$$

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Here R is chosen so that $D \subset B_R = \{x: |x| \leq R\}$, $h_p(kr)$ are the spherical Hankel functions normalised so that $h_p(kr) \sim r^{-1} \exp(ikr)$ as $r \rightarrow \infty$ that is

$$h_p(kr) = k(\pi/2kr)^{1/2} H_{p+1/2}^{(1)}(kr) \exp[\frac{1}{2}i\pi(p+1)]$$

and v is defined in (3). Formula (6) holds because its right-hand side solves (4) and (1) in $\Omega_R = R^3 \setminus B_R$, and there is at most one function with these properties. Indeed, if v and v_1 solve (1) in B_R and (4), then $w \equiv v - v_1$ solves (1) and $w = o(r^{-1})$ as $r \rightarrow \infty$. According to the well known result, $w \equiv 0$ in Ω_R (see, e.g., lemma 1.2.1 in [1, p 25]).

According to (3),

$$\exp(ikn \cdot \nu r(\nu)) + v(\nu, r(\nu)) = 0 \quad \forall \nu \in S^2. \quad (7)$$

Therefore let us look for the function $r = r(\nu)$ which solves the minimisation problem

$$F(r(\nu)) \equiv \int_{S^2} d\nu \left| \exp(ikn \cdot \nu r(\nu)) + \sum_{p=0}^{\infty} f_p Y_p(\nu) h_p(kr(\nu)) \right|^2 = \text{minimum}. \quad (8)$$

Since the coefficients f_p are known if the function $f(\nu) = f(n, \nu, k)$ is known for all $\nu \in S^2$, problem (8) can be solved in principle numerically, e.g. by a Newton-type method.

The functional F defined by (8) is a continuous mapping from $C(S^2)$ into $R_+ = \{x: x \geq 0\}$, where $C(S^2)$ is the space of continuous functions on S^2 with the usual norm. Consider a compact M in $C(S^2)$ defined by

$$M = \{r(\nu): 0 < a \leq r(\nu) \leq b, \sup_{\nu \in S^2} [r(\nu) + |\nabla r(\nu)|] \leq A\} \quad (9)$$

where a , b and A are some given positive constants that contain *a priori* information about Γ , and ∇ in (9) is the gradient on S^2 .

Let us assume that $\Gamma \in M$, i.e. if $r = r(\nu)$ is the equation of Γ then $r(\nu) \in M$. Since $F: C(S^2) \rightarrow R_+$ is continuous and M is compact we conclude that F attains its minimum on M . By the assumption $r(\nu) \in M$ and $F(r(\nu)) = 0$ so that $r = r(\nu)$, the equation of Γ , minimises $F(r(\nu))$ and this minimum is equal to zero.

It is not proved that the scattering data $f(\nu)$ determine Γ uniquely (see [1, p 399]). Therefore we do not know if the minimiser, i.e. the solution to (8), is unique. It is a longstanding open problem to prove or disprove that the data $f(\nu) = f(n, \nu, k)$ (with n and k fixed and ν running through S^2) determine Γ uniquely.

Nevertheless the suggested method for solving the inverse problem can be tried numerically. If Γ is a sphere one can easily use the given method for exact recovery of Γ from $f(\nu)$.

Remark. It is known that the series (6) may diverge at some points on Γ (see [1, p 403] for some references). We do not know how this influences the inversion procedure based on (8). It is known that the system $\{Y_p(\nu) h_p(kr(\nu))\}$, $p = 0, 1, 2, \dots$ is closed in $L^2(\Gamma)$ (see e.g. [1, p 162]), i.e. if $f \in L^2(\Gamma)$ then

$$\int_{\Gamma} f Y_p(\nu) h_p(kr(s)) ds = \int_{S^2} f(r(\nu)) Y_p(\nu) h_p(kr(\nu)) a(\nu) d\nu = 0 \quad \forall p$$

implies $f = 0$, where $a(\nu) = ds/d\nu \geq 0$, ds is the element of the surface area of Γ and $d\nu$ is the element of the surface area of S^2 .

Finally we note that the method given in this Letter remains valid for other boundary conditions on Γ , e.g. for the Neumann boundary condition and for the impedance

boundary condition $u_N + \eta u = 0$ on Γ , where u_N is the normal derivative of u on Γ and η is a given constant.

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Reference

- [1] Ramm A G 1986 *Scattering by Obstacles* (Dordrecht: Reidel)