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LETTER TO THE EDITOR

An inverse problem for the Helmholtz equation in a semi-infinite medium

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Abstract. Let $\Delta u+\omega^2 n(z)u=-\delta(x-y)$, $x=(x_1,x_2,x_3)$, $x_3=z<0$. Let $x^1=(x_1,x_2,0)$. Assume that $u(x^1,y^1,\omega)$ is known for all x^1, fixed $y^1=(0,0,0)$, and $\omega_0>\omega>0$, where $\omega_0>0$ is a small fixed number. We find $n(z)$ from these data.

Let

$$\Delta u+\omega^2 n(z)u=-\delta(x)\qquad z=x_3<0,\quad \omega>0. \tag{1}$$

Taking the Fourier transform of (1) in x^1 yields

$$\tilde{u}''-\lambda^2\tilde{u}+q(z)\tilde{u}=-\delta(z)\qquad q(z)=\omega^2 n(z) \tag{2}$$

where

$$\tilde{u}(\lambda,z)=\int_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda x^1)u\,\mathrm{d}x^1\qquad x^1=(x_1,x_2,0)\qquad \lambda=(\lambda_1,\lambda_2).$$

Let

$$\lambda^2>q_0=\omega^2\max n(z). \tag{3}$$

Then the integral equation

$$\tilde{u}=g+\int_{-\infty}^{\infty}g(z-z')q\tilde{u}\,\mathrm{d}z'\qquad g=\frac{\exp(-\lambda|z|)}{2\lambda} \tag{4}$$

is uniquely solvable by iterations and $\tilde{u}$ is analytic in λ in the region $\operatorname{Re}\lambda>q_0^{1/2}$. If $\omega\to 0$ then (4) implies

$$f(z,\lambda)=\lim_{\omega\to 0}\frac{\tilde{u}-q}{\omega^2}=\int_{-\infty}^{\infty}\frac{\exp(-\lambda|z-z'|)}{2\lambda}n(z')\frac{\exp(-\lambda|z'|)}{2\lambda}\,\mathrm{d}z'. \tag{5}$$

This low frequency limit was used in [1, 2, 3]. Our data are $\tilde{u}(\lambda,0)$, therefore the function $f(0,\lambda)$ is known. Let $4\lambda^2 f(0,\lambda)=F(\lambda)$. Then

$$F(\lambda)=\int_{-\infty}^{\infty}\exp(-2\lambda|z'|)n(z')\,\mathrm{d}z'. \tag{6}$$

We assume that $n(z)=1$ for $z>0$ (in air) and $n(z)=n_0=\text{constant}$ for $z<-d$ (deep in the earth), where d is a certain depth. Then (6) becomes

$$\int_{-d}^{0}\exp(-2\lambda|z|)n(z)\,\mathrm{d}z=\psi(\lambda) \tag{7}$$

where

$$\psi(\lambda)\equiv F(\lambda)-\int_0^\infty \exp(-2\lambda z)\,\mathrm{d}z-n_0\int_{-\infty}^{-d}\exp(-2\lambda|z|)\,\mathrm{d}z$$
$$=F(\lambda)-\frac{1}{2\lambda}-\frac{n_0\exp(-2\lambda d)}{2\lambda}. \tag{8}$$

A change of variables, $2\lambda=p$, $z=-t$, transforms (7) into

$$\int_0^d \exp(-pt)h(t)\,\mathrm{d}t=\varphi(p)$$
$$h(t)\equiv n(-t), \qquad \varphi(p)\equiv\psi(\tfrac{1}{2}p). \tag{9}$$

The function $h(t)$ can be found by the method given in [2, 4]. Namely, define

$$h_N(t)=\int_0^b \varphi(p)H_N(p-\tfrac{1}{2}b)\exp(pt)\,\mathrm{d}p \tag{10}$$

where $b>0$ is an arbitrary fixed number,

$$H_N(p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta_N(\mathrm{i}y)\exp(-\mathrm{i}yp)\,\mathrm{d}y$$
$$\delta_N(\mathrm{i}y)=\left(\frac{N}{4\pi d^2}\right)^{1/2}\left(1+\frac{y^2}{4d^2}\right)^N\left(\frac{\sin[yb/(4N+2\eta)]}{yb/(4N+2\eta)}\right)^{2N+\eta} \tag{11}$$

where $\eta\geqslant 1$ is an arbitrary number. Then $\|h_N-h\|\to 0$ as $N\to\infty$, where the norm is $L^2([0,d])$ $(C([0,d]))$ norm if $h\in L^2([0,d])$ $(C([0,d]))$, $n(z)=h(-z)$, $0>z>-d$.

Assume that $n(z)=1$ if $z>0$, $n(z)=n_0$ if $z<-d$, $n(z)=n_j$ if $z_{j+1}>z>z_j$, $1\leqslant j\leqslant m$, $z_1=-d$, $z_{m+1}=0$, so that one has m homogeneous layers in the region $0>z>-d$. Then equation (7) can be written as

$$\sum_{j=1}^{m} n_j\,\frac{\exp(-2\lambda|z_{j+1}|)-\exp(-2\lambda|z_j|)}{2\lambda}=\psi(\lambda). \tag{12}$$

One can take m values of λ and find n_j, $1\leqslant j\leqslant m$, from the resulting linear system for n_j.

Example. Let $m=1$ (one layer). Then $z_1=-d$, $z_2=0$, $n_1=2\lambda\psi(\lambda)[1-\exp(-2\lambda d)]^{-1}$.

It is natural to assume that we know $n(z)$ for $z>0$, $n(z)=1$ (in air). If we do not assume that $n(z)=n_0$ for $z<-d$ then (6) gives the Laplace transform of $h(t)=n(-t)$, $t>0$:

$$\int_0^\infty \exp(-2\lambda t)h(t)\,\mathrm{d}t=F(\lambda)-\frac{1}{2\lambda} \qquad \lambda>0. \tag{13}$$

The function $h(t)$ can be found from (13) numerically. If $h(\infty)=n_0=\text{constant}$, then

$$n_0=\lim_{\lambda\to 0}(2\lambda F(\lambda)-1)=-1+2\lim_{\lambda\to 0}\lambda F(\lambda). \tag{14}$$

If a small current loop (a magnetic dipole) is placed on the surface of the earth and $\mu=$ constant, $\varepsilon'=\varepsilon+\mathrm{i}\sigma/\omega$, $\varepsilon'=\varepsilon'(\rho,z)$, where $\rho=(z_1^2+z_2^2)^{1/2}$, $z=z_3$, the dipole is at the origin of the coordinate system, μ, ε and σ are the magnetic and dielectric parameters and conductivity of the earth, $\sigma=0$ if $z>0$, then the electric field is $\boldsymbol{E}=u(\rho,z)\boldsymbol{e}_\varphi$, where $\boldsymbol{e}_\varphi$ is the unit vector of the cylindrical coordinate system.

The scalar function u satisfies the equation

$$\Delta u - u/\rho^2 + \omega^2 \varepsilon' \mu u = -\mathrm{i}\omega\mu J \delta(\rho - \alpha)\delta(z). \tag{15}$$

Indeed,

$$\nabla \times \boldsymbol{E} = \mathrm{i}\omega\mu \boldsymbol{H} \qquad \nabla \times \boldsymbol{H} = -\mathrm{i}\omega\varepsilon' \boldsymbol{E} - J\delta(\rho - \alpha)\delta(z)\boldsymbol{e}_\varphi \qquad J = m/\pi\alpha^2$$

$$\nabla \cdot \boldsymbol{H} = 0 \qquad \nabla \cdot (\varepsilon' \boldsymbol{E}) = 0.$$

Here m is the strength of the magnetic dipole, $m =$ constant. We have $\nabla \times \nabla \times \boldsymbol{E} = \mathrm{i}\omega\mu J\delta(\rho - \alpha)\delta(z)\boldsymbol{e}_\varphi + \omega^2\varepsilon'\mu\boldsymbol{E}$. If $\boldsymbol{E} = u(\rho, z)\boldsymbol{e}_\varphi$, then $\nabla \cdot \boldsymbol{E} = 0$ and therefore (15) holds. Define the Bessel transform $\tilde{f} = Bf = \int_0^\infty \mathrm{d}\rho\, \rho J_1(\lambda\rho) f(\rho)$, where $J_p(x)$ is the Bessel function. Taking this transform of equation (15) and assuming that ε' does not depend on ρ, one obtains

$$\frac{\mathrm{d}^2 \tilde{u}}{\mathrm{d}z^2} + (\omega^2 \varepsilon'(z)\mu - \lambda^2)\tilde{u} = -\mathrm{i}\omega\mu \frac{m}{2\pi} \lambda\delta(z). \tag{16}$$

If $u(\rho, 0, \omega)$ is measured, then $\tilde{u}(\lambda, 0, \omega)$ is known. These data allow one to find $\varepsilon'(z)$ by the method described earlier. Namely

$$-4\lambda^2 \lim_{\omega \to 0} \frac{\tilde{u} - \mathrm{i}\omega\lambda M g(z, \lambda)}{\omega^2 \mu\lambda M} = \int_{-\infty}^{\infty} \exp[-\lambda(|z - z'| + |z'|)]\sigma(z')\, \mathrm{d}z' \tag{17}$$

where $M = \mu m/2\pi$ and g is defined in (4). At $z = 0$ the left-hand side of (17) is known since $\tilde{u}$ is known at $z = 0$. Take $z = 0$ in (17) to get the equation of the form (6) for $\sigma(z)$. Since $\sigma(z) = 0$ for $z > 0$, this equation yields the Laplace transform of $\sigma(z)$, and inverting this Laplace transform one finds $\sigma(z)$. If $\sigma(z)$ is found one can find $\varepsilon(z) = \operatorname{Re} \varepsilon'(z)$ using a similar method.

If $\varepsilon = \varepsilon(z, \omega)$ then one finds $\varepsilon(z, 0) = \lim_{\omega \to 0} \varepsilon(z, \omega)$.

Equation (17) for $z = 0$ is of the form

$$-4\lambda \lim_{\omega \to 0} \frac{\tilde{u}(\lambda, 0) - \frac{1}{2}\mathrm{i}\omega M}{\omega^2 \mu M} = \int_{-\infty}^{0} \exp(-2\lambda|z'|)\sigma(z')\, \mathrm{d}z'. \tag{18}$$

The equation for $\varepsilon(z)$ is of the form

$$\lim_{\omega \to 0} 4\lambda \frac{\tilde{u}(\lambda, 0) - \frac{1}{2}\mathrm{i}\omega M + (\omega^2 \mu M/4\lambda) \int_{-\infty}^{0} \exp(-2\lambda|z|)\sigma(z)\, \mathrm{d}z - w}{\mathrm{i}\omega^3 \mu M}$$

$$= \int_{-\infty}^{\infty} \exp(-2\lambda|z'|)\varepsilon(z')\, \mathrm{d}z'. \tag{19}$$

where $w := \mathrm{i}^3 \omega^3 \mu^2 M (8\lambda^2)^{-1} \int_{-\infty}^{0} \exp(-\lambda|z|)\sigma(z) \{\int_{-\infty}^{0} \sigma(z') \exp[-\lambda(|z - z'| + |z'|)]\, \mathrm{d}z'\}\, dz$. Since $\varepsilon(z) = \varepsilon_0$, $z > 0$, is known, equation (19) yields the Laplace transform of $\varepsilon(z)$, $z < 0$. Inverting this Laplace transform one finds $\varepsilon(z)$.

The problems we discuss are ill-posed and practical implementation of the method is not trivial. A discussion of the numerical inversion of the Laplace transform and the references on this subject are given in [2].

Remark. Equation (6) was derived under the assumption $0 < n(z) \leqslant c$, $-\infty < z < \infty$, but holds under the weaker assumption

$$|n(z)| \leqslant c(1 + |z|)^m \qquad \operatorname{Im} n(z) \geqslant 0 \qquad -\infty < z < \infty \tag{20}$$

where c and m are arbitrary fixed positive numbers.

The derivation of (6) under assumption (20) requires a new idea. First consider equation (2ε), i.e. (2) with $\lambda_\varepsilon^2 \equiv \lambda^2 - \mathrm{i}\varepsilon$, $\varepsilon > 0$, in place of λ^2. If $\operatorname{Im} n(z) = 0$, then the expression $\mathrm{d}^2/\mathrm{d}z^2 - q(z)$ in (2) defines a self-adjoint operator in $H = L^2(-\infty, \infty)$, equation (2ε) is uniquely solvable in H, and if $\tilde{u}(\lambda_\varepsilon, z, \omega)$ is its solution then the limit (5) exists with λ_ε in place of λ. The limit is understood now as the $C_{\mathrm{loc}}(-\infty, \infty)$ limit, i.e. uniform pointwise convergence on compact sets of the real line. After taking this limit one can pass to the limit $\varepsilon \to 0$ and obtain (5). If $\operatorname{Im} n(z) \geqslant 0$ and (20) holds, an additional argument is needed to establish the existence of the solution to (2ε) for any $\varepsilon > 0$.

The argument is as follows. Under assumption (20) the operator in equation (2ε) is of the form $A + \mathrm{i}B$ where A and B are self-adjoint operators in H and $(Bu, u) \geqslant \varepsilon(u, u)$ on $\mathscr{D}(A) \cap \mathscr{D}(B)$. Here A is defined by the differential expression $\mathrm{d}^2/\mathrm{d}z^2 - \lambda^2 + \omega^2 \operatorname{Re} n(z)$ and B is the operator of multiplication by $\varepsilon + \omega^2 \operatorname{Im} n(z) \geqslant \varepsilon$. The operator $A + \mathrm{i}B$ is boundedly invertible in H. Indeed, one can easily check that its range is closed, its null space is trivial and the null space of the adjoint operator $A - \mathrm{i}B$ is trivial; all these conclusions follow from the inequality $(Bu, u) \geqslant \varepsilon(u, u)$. Thus, for any $\varepsilon > 0$, the equation (2ε) is uniquely solvable in H under the assumption (20). The rest of the argument is the same as above: first pass to the limit $\omega \to 0$ and then take $\varepsilon \to 0$ to obtain (5). Equation (6) follows from (5) immediately.

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