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LETTER TO THE EDITOR

Inversion of the Laplace transform

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Abstract. Let

$$\int_0^b \exp(-pt)f(t)\,\mathrm{d}t = F(p) \qquad p>0$$

where $b>0$ is a given number. We give a formula for $f(t)$ in terms of $F(p)$, $p>0$.

The object of this Letter is to give a formula for the inverse Laplace transform in which the data used are given on the semi-axis $p>0$ only.

Let

$$\int_0^b \exp(-pt)f(t)\,\mathrm{d}t = F(p) \qquad p>0 \tag{1}$$

where $b>0$ is a fixed number. We have (1, formula 4.14.1)

$$\int_0^\infty \exp(-pt)J_0(px)\,\mathrm{d}p = (t^2+x^2)^{-1/2}. \tag{2}$$

where J_0 is a Bessel function. It follows from (1) and (2) that

$$\int_0^b \mathrm{d}t\,(t^2+x^2)^{-1/2}f(t) = F_1(x) \qquad x>0 \tag{3}$$

where

$$F_1(x) = \int_0^\infty F(p)J_0(px)\,\mathrm{d}p. \tag{4}$$

Let $t^2=\tau$, $x^2=\xi$, $F_1(\sqrt{\xi})=F_2(\xi)$, $(2\tau^{1/2})^{-1}f(\tau^{1/2})=f_1(\tau)$. Then (3) can be written as

$$\int_0^{b^2} \frac{f_1(\tau)\,\mathrm{d}\tau}{(\tau+\xi)^{1/2}} = F_2(\xi) \qquad \xi>0. \tag{5}$$

Let $\tau=b^2s^{-1}$, $\xi=b^2q^{-1}$, $f_1(b^2s^{-1})bs^{-3/2}=f_2(s)$, $q^{-1/2}F_2(b^2q^{-1})=F_3(q)$. Then (5) can be written as

$$\int_1^\infty \frac{f_2(s)\,\mathrm{d}s}{(s+q)^{1/2}} = F_3(q) \qquad q>0. \tag{6}$$

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Let $q=\cosh z\equiv\frac{1}{2}(e^{z}+e^{-z})$. Multiply (6) by $\cos(zy)$ and integrate in z over $(0,\infty)$ to get

$$\int_1^\infty f_2(s)\int_0^\infty \frac{\cos(zy)\,dz}{(s+\cosh z)^{1/2}}=\int_0^\infty F_3(\cosh z)\cos(zy)\,dz\equiv F_4(y)\qquad y\geqslant 0. \tag{7}$$

Let us use the equation [2, formulae (7.4.1) and (7.4.6)]

$$\int_0^\infty \frac{\cos(zy)\,dz}{(\cosh x+\cosh z)^{1/2}}=\frac{1}{\cosh(\pi y)}\int_0^x \frac{\cos(zy)\,dz}{(\cosh x-\cosh z)^{1/2}}. \tag{8}$$

Formula (8) follows also from the relations used in the theory of Mehler–Fock transforms [3]:

$$\frac{\sqrt{2}}{\pi}\int_0^x \frac{\cos(yz)\,dz}{(\cosh x-\cosh z)^{1/2}}=P_{-1/2+iy}(\cosh x)=\frac{\sqrt{2}}{\pi}\cosh(\pi y)\int_0^\infty \frac{\cos(yz)\,dz}{(\cosh x+\cosh z)^{1/2}}. \tag{9}$$

It follows from (7) and (8), with $s=\cosh x$, $f_3(x)=f_2(\cosh x)\sinh x$ and $F_5(y)=F_4(y)\cosh(\pi y)$, that

$$\int_0^\infty dx f_3(x)\int_0^x \frac{\cos(zy)\,dz}{(\cosh x-\cosh z)^{1/2}}=F_5(y)\qquad y\geqslant 0,\quad F_5=F_4(y)\cosh(\pi y). \tag{10}$$

Change the order of integration to get

$$\int_0^\infty dz\cos(zy)\int_z^\infty \frac{f_3(x)\,dx}{(\cosh x-\cosh z)^{1/2}}=F_5(y)\qquad y\geqslant 0. \tag{11}$$

Take the inverse cosine transform to obtain

$$\int_z^\infty \frac{f_3(x)\,dx}{(\cosh x-\cosh z)^{1/2}}=F_6(z)\qquad z\geqslant 0 \tag{12}$$

where

$$F_6(z)=\frac{2}{\pi}\int_0^\infty F_5(y)\cos(zy)\,dy. \tag{13}$$

Let $\cosh x=u^{-1}$, $\cosh z=v^{-1}$, $u^{-1/2}f_3(\cosh^{-1}u^{-1})(1-u^2)^{-1/2}=f_4(u)$ and $F_6(\cosh^{-1}v^{-1})v^{-1/2}=F_7(v)$. Then (12) can be written as an Abel equation

$$\int_0^v \frac{f_4(u)\,du}{(v-u)^{1/2}}=F_7(v)\qquad 1\geqslant v>0. \tag{14}$$

This equation is solvable in the closed form. Namely, if

$$\int_0^v \frac{f(u)\,du}{(v-u)^{\mu}}=F(v)\qquad 0<\mu<1 \tag{15}$$

then

$$f(u)=\frac{\sin(\mu\pi)}{\pi}\frac{d}{du}\int_0^u \frac{F(v)\,dv}{(u-v)^{1-\mu}}, \tag{16}$$

see, e.g. [4]. Thus, (14) implies

$$f_4(u)=\frac{1}{\pi}\frac{d}{du}\int_0^u \frac{F_7(v)\,dv}{(u-v)^{1/2}}. \tag{17}$$

Given $f_4(u)$ one finds

$$f_3(\cosh^{-1} u^{-1}) = u^{1/2}(1-u^2)^{1/2} f_4(u) \qquad f_3(x) = f_4(1/\cosh x)(\cosh x)^{-3/2} \sinh x$$

$$f_2(\cosh x) = f_4(1/\cosh x)(\cosh x)^{-3/2} \qquad f_2(s) = f_4(1/s)s^{-3/2}$$

$$f_1(b^2 s^{-1}) = b^{-1} s^{3/2} s^{-3/2} f_4(1/s) = b^{-1} f_4(1/s)$$

$$f_1(\tau) = b^{-1} f_4(\tau/b^2)$$

$$f(t) = 2t f_1(t^2) = 2tb^{-1} f_4(t^2/b^2).$$

Thus

$$f(t) = \frac{2tb^{-1}}{\pi} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \frac{F_7(v)\,\mathrm{d}v}{(u-v)^{1/2}} \bigg|_{u=t^2 b^{-2}}. \tag{18}$$

Here

$$F_7(v) = v^{-1/2} \frac{2}{\pi} \int_0^\infty \mathrm{d}y \cos(y \cosh^{-1} v^{-1}) \cosh(\pi y)$$
$$\times \int_0^\infty \mathrm{d}z \cos(zy)(\cosh z)^{-1/2} \int_0^\infty \mathrm{d}p\, F(p) J_0\left(p \frac{b}{(\cosh z)^{1/2}}\right). \tag{19}$$

The inversion formula (18) is of interest since it uses only the data $F(p)$ on the real line and there is no need to continue $F(p)$ analytically onto the Mellin contour, as in the usual inversion formula.

Our calculations are similar to those in [5], where formula (8) was used and an Abel equation was also obtained. The inversion formulae (18)–(19) here differ from the formulae in [5], where some other changes of variables were used. In [6] inversion of the Laplace transform from a finite segment $0 < p < p_0$ is given. In [7] a number of the inverse geophysical problems are studied. Inversion of the Laplace transform from the real axis is a part of the solution to these problems.

Example. Let us take $b = 1$ and $f = \delta(t)$ in (1), where $\delta(t)$ is the delta function. Then $F(p) = 1$. Since

$$\int_0^\infty \mathrm{d}p F(p) J_0[p(\cosh z)^{-1/2}] = (\cosh z)^{1/2}$$

and

$$\frac{2}{\pi} \int_0^\infty \mathrm{d}z \cos(zy) = \delta(y)$$

formula (19) yields $F_7(v) = v^{-1/2}$. Substitute this in (18) to obtain

$$f(t) = \frac{2t}{\pi} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \frac{\mathrm{d}v}{v^{1/2}(u-v)^{1/2}} \bigg|_{u=t^2} = \frac{2t}{\pi} \frac{\mathrm{d}\varphi(u)}{\mathrm{d}u} \bigg|_{u=t^2}$$

where $\varphi(u)=\pi$ if $u>0$, $\varphi(u)=0$ if $u<0$. Since $\mathrm{d}\varphi/\mathrm{d}u=\pi\delta(u)$, one has $f(t)=2t\delta(t^2)=\delta(t)$. Here one takes into account that

$$\int_0^\infty \psi(t)2t\delta(t^2)\,\mathrm{d}t=\int_0^\infty \psi(\sqrt{x})\delta(x)\,\mathrm{d}x=\psi(0)$$

for any continuous function $\psi(t)$ defined on $[0, \infty)$ and vanishing at infinity.

Inversion of the Laplace transform has been studied in many papers (see, e.g., [8], [11 and references therein], [9, 10, 12–14]). The problem of the Laplace inversion from the real axis was studied in [9, p 316], where an inversion formula based on the Mellin transform was used. However, this formula requires analytic continuation of the Mellin transform into the complex plane and in this respect is of the same type as the usual inversion formula, which requires analytic continuation of the Laplace transform from the real axis onto the Mellin contour. Widder [10] gave an inversion formula which uses the Laplace transform on the real axis only, but one has to differentiate the data infinitely many times, so that this inversion is not convenient for practical purposes. In [12] a numerical inversion based on eigenfunction expansions is suggested. In [13] and [14] some Post–Widder-type operators and more general operators are studied. The formula given here is new and our idea is close to the idea in [5], where the inversion was also reduced to an Abel-type integral equation. Our final inversion formula (18) differs from the formula given in [5] because the reduction of the inversion problem to the Abel equation is different in [5]. Examples of the applicability of the inversion formula are not given in [5].

Finally we should emphasise that the inversion of the Laplace transform is a highly ill-posed problem. Clearly, for the solvability of equation (1) it is necessary that $F(p)$ be an entire function of p of exponential type b. If one would like to give a necessary and sufficient condition for the solvability of equation (1), one might say that equation (1) is solvable in, say, $\mathrm{L}^p(0, b)$ iff the inversion procedure by formula (18) gives a function $f \in \mathrm{L}^p(0, b)$ which vanishes outside $(0, b)$.

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