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LETTER TO THE EDITOR

Inverse scattering: asymptotic analysis

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Abstract. If an acoustic field is governed by the equation $\nabla^2 u + \omega^2 u + \omega^2 a_1(x)u + \nabla \cdot (a_2(x)\nabla u) = -\delta(x-y)$ in R^3 and u is measured on the surface of the earth, i.e. on the plane $x_3 = 0$ for all positions of the source y and receiver x and for all frequencies, then: (i) we show that the low frequency portion of the data determines $a_2(x)$ while the high frequency portion of the data determines $(a_1 - a_2)(1 + a_2)^{-1} \equiv p(x)$. Here $a_2 = \rho^{-1}\Delta\rho$ is the relative variation of the density of the inhomogeneity and $p(x)$ depends *only* on the relative variation of the velocity in the inhomogeneity; (ii) we show how to recover a_2 from the low frequency data and how to recover $p(x)$ from the high frequency data; (3) we give a formula for the high frequency asymptotics of u on the plane $x_3 = 0$.

In [1]–[3] an exact solution to some three-dimensional inverse problems of geophysics is given. The problem we discuss here is the following one. Let u be the acoustic field generated in an inhomogeneous medium by a point source:

$$\nabla^2 u + \omega^2 u + \omega^2 a_1(x)u + \nabla \cdot (a_2(x)\nabla u) = -\delta(x-y) \text{ in } R^3. \tag{1}$$

Here $a_1 = K^{-1}\Delta K$, $a_2 = \rho^{-1}\Delta\rho$ are the relative variations of the bulk modulus and density, ω is the frequency, $\delta(x)$ is the delta function. Assume that $a_1(x)$ and $a_2(x)$ are compactly supported in $R^3_- = \{x : x_3 < 0\}$, and $1 + a_2 > 0$. This means that the inhomogeneity lies in the lower half-space. The data are the values of $u(x, y, \omega)$ measured on the plane $P = \{x : x_3 = 0\}$ for all positions of the source $y^1 = (y_1, y_2, 0)$ and the receiver $x^1 = (x_1, x_2, 0)$ and for all $\omega > 0$. The problem is to recover $a_j(x)$, $j = 1, 2$, from the data. We show that the low frequency portion of the data determines a_2 and the high frequency portion determines $p(x) = (1 + a_2)^{-1}(a_1 - a_2)$. Thus, by comparing two sets of data with the same high frequency portions of the data one concludes without calculations that the two sets of the data correspond to the media with different densities. One notes that $p(x) = -(1 + a_3)^{-2}(2a_3 + a_3^2)$ where $a_3 \equiv v^{-1}\Delta v$ is the relative change in the velocity, $v = (\rho^{-1}K)^{1/2}$. If the low frequency portions of the data are the same, then one concludes that the difference in the data is due to the variation of the velocity. We give some methods for finding $a_2(x)$ and $p(x)$ separately from the low and high frequency data respectively and we give an asymptotic formula for u on P for large ω.

If $\omega \to 0$ then the limit equation (1) is of the form

$$\nabla^2 G_0 + \nabla \cdot (a_2(x)\nabla G_0) = -\delta(x-y) \tag{2}$$

and $u(x, y, \omega) \to G_0(x, y)$ as $\omega \to 0$. The existence of this limit follows from the results in [3, § V.4] provided that $1 + a_2 > 0$. Let $g_0 = (4\pi|x-y|)^{-1}$. Then (2) can be written as

$$G_0 = g_0 + \int g_0 \nabla \cdot (a_2 \nabla G_0)\, \mathrm{d}z \tag{3}$$

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where the integral is taken over the support of a_2. If a_2 and ∇a_2 are small, then one puts $G_0 = g_0$ under the sign of the integral in (3) and after integrating by parts one obtains a linear equation for a_2:

$$\int \nabla_z g_0(x^1, z) \cdot \nabla_z g_0(z, y^1) a_2 \, \mathrm{d}z = f(x^1, y^1) \qquad f(x^1, y^1) = g_0(x^1, y^1) - G_0(x^1, y^1). \tag{4}$$

Equation (4) can be solved analytically for a_2 by the method given in [1]. One can solve (4) numerically using a regularisation scheme. E.g., denote the integral operator in (4) by M, the noisy data by f_δ, $\|f_\delta - f\| \leqslant \delta$, where $\delta > 0$ is known ($\|f\| = \|f\|_{L^2(P\times P, w)}$, where $w(x)$ is a weight function, e.g. $w = (1 + |x^1| + |y^1|)^{-\alpha}$, $\alpha > 3$, or $w = 1$ if $|x^1|^2 + |y^1|^2 \leqslant R^2$, $w(x) = 0$ if $|x^1|^2 + |y^1|^2 > R^2$), and consider the variational problem:

$$\|Mv - f_\delta\|^2 + \alpha \|v\|_1^2 = \min. \tag{5}$$

Here $\|v\|_1^2 = \int_{B_R} (|v|^2 + |\nabla v|^2) \, \mathrm{d}x$, $B_R = \{x : |x| \leqslant R\}$ contains the support of a_2, and $\alpha = \alpha(\delta) > 0$ is chosen so that $\alpha(\delta) \to 0$, $\delta^2 \alpha^{-1}(\delta) \to 0$ as $\delta \to 0$. Assume that $\|a_2\|_1 < \infty$. Then problem (5) is uniquely solvable and its solution $v_\delta \to a_2$ as $\delta \to 0$.

If $\omega \to +\infty$ then (1) can be written as

$$\nabla^2 u + \omega^2 u + \omega^2 p(x) u = -\delta(x - y^1) \qquad p(x) = (1 + a_2)^{-1}(a_1 - a_2). \tag{6}$$

Here we have used the equation $(1 + a_2)^{-1}\delta(x - y^1) = \delta(x - y^1)$ which holds since $y^1 \notin \operatorname{supp} a_2$ (remember that $y^1 \in P$ and P does not intersect $\operatorname{supp} a_j, j = 1, 2$), and the term $(1 + a_2)^{-1} \nabla a_2 \cdot \nabla u$ was dropped since in the following we only use the time of propagation, and the input of this term to the time of propagation is negligible as will be shown in the high frequency analysis.

Let us write the high frequency asymptotics of the solution to (6) on the plane P. Assume that $p(x)$ is n times continuously differentiable, i.e. $p(x) \in C^n$, but $p(x) \notin C^{n+1}$. Assume that singsupp consists of finitely many smooth surfaces and that the normal derivative $\partial_\nu^{n+1} p$ jumps across these surfaces. Here ν is the unit normal to singsupp p. Suppose that the straight line joining x and y does not intersect supp p and there exist finitely many points $\xi^{(j)} \in$ singsupp p, $1 \leqslant j \leqslant m$, such that the vector $(\nabla_\xi \varphi(\xi, y) + \nabla_\xi \varphi(x, \xi))_{\xi = \xi^{(i)}}$ does not vanish and is orthogonal to singsupp p at $\xi^{(i)}$. Here $\varphi(x, \xi) \geqslant 0$ is the phase of the asymptotics of u, $u(x, y, \omega) \simeq [\exp(\mathrm{i}\omega\varphi(x, y))/4\pi\varphi(x, y)] z_0(x, y) + \mathrm{O}(\omega^{-1})$, $\omega \to +\infty$, $z_0(x, x) = (1 + p(x))^{1/2}$, $\varphi(x, x) = 0$, $|\nabla_y \varphi(x, y)|^2 = 1 + p(y)$. Let $\zeta = \zeta^{(i)}(\eta)$ be a parametric equation of singsupp p in a neighbourhood of $\xi^{(i)}$, $\zeta^{(i)}(0) = \xi^{(i)}$, $1 \leqslant i \leqslant m$. Assume that $|\det \zeta_a^{(i)}(0) \zeta_b^{(i)}(0)| = 1$, where $\zeta_a^{(i)}(0) = \partial \zeta^i(0)/\partial \eta_a$, $a, b = 1, 2$. Let $\psi^{(i)}(\eta) = \varphi(x, \zeta^{(i)}(\eta)) + \varphi(\zeta^{(i)}(\eta), y)$. Assume that $\det \psi^{(i)}(0)'' \neq 0$, $1 \leqslant i \leqslant m$. This assumption implies that the point x is not on a caustic of the rays issued from y and reflected at singsupp p in a neighbourhood of $\xi^{(i)}$.

Theorem 1. Under the above assumptions the following formula holds

$$u(x, y, \omega) = \frac{\exp(\mathrm{i}\omega|x - y|)}{4\pi|x - y|} + \frac{A}{\omega^{n+1}} + \mathrm{O}\left(\frac{1}{\omega^{n+2}}\right) \qquad \text{as } \omega \to +\infty \tag{7}$$

where

$$A = \sum_{i=1}^{m} \frac{z_0(x, \xi^{(i)}) z_0(\xi^{(i)}, y) \exp(\mathrm{i}\omega\psi^{(i)}(0))}{4\pi(\det \psi''(\xi^{(i)}))^{1/2} \varphi(x, \xi^{(i)}) \varphi(\xi^{(i)}, y)} \frac{\mathrm{i}^{n+1}}{[2(1 + p(\xi^{(i)}))^{1/2} \cos\gamma]^{n+2}}$$
$$\times [\partial_{\nu(\xi^{(i)})}^{n+1} p(\xi^{(i)}+) - \partial_{\nu(\xi^{(i)})}^{n+1} p(\xi^{(i)}-)] \tag{8}$$

$\nu(\xi)=\nabla_\xi[\varphi(\xi,y)+\varphi(x,\xi)]/|\nabla_\xi[\varphi(\xi,y)+\varphi(x,\xi)]|$, γ is the angle between $\nu(\xi)$ and $\nabla_\xi\varphi(\xi,y)$, and $\partial^{n+1}_{\nu(\xi^{(i)})}p(\xi^{(i)}\pm)=\lim_{\varepsilon\downarrow 0}\partial^{n+1}_{\nu(\xi^{(i)})}p(\xi^{(i)}\pm\varepsilon\nu(\xi^{(i)}))$.

The proof of this theorem uses the WKB ansatz

$$u_l(x,y,\omega)=\sum_{m=0}^{l}\omega^{-2m}H_{1/2-m}(\omega\varphi(x,y))z_m(x,y) \qquad H_\nu(t)=\tfrac{1}{4}(2\pi)^{-1/2}\mathrm{i}\omega t^{-\nu}H^{(1)}_\nu(t) \tag{9}$$

where $H^{(1)}_\nu(t)$ is the Hankel function. The functions φ and z_m are determined by the eikonal and transport equations and initial conditions $\varphi(x,x)=0$, $z_0(x,x)=(1+p(x))^{1/2}$, $z_m(x,y)=\mathrm{O}(1)$, $y\to x$. In [4] there is a justification of this ansatz and estimates of $u-u_l$ as $\omega\to\infty$ for a different equation. In [5] an ansatz similar to (9) was used for a formal construction of the high frequency asymptotics of the point source in an inhomogeneous medium.

We establish the formula

$$u(x,y,\omega)=\sum_{m=0}^{\infty}Q_l^m u_l(x,y,\omega), \tag{10}$$

where

$$Q_l f=\omega^{-2l}\int_{R^3}H_{1/2-l}(\omega\varphi(x,y))\Delta_y z_l(x,y)f(y)\,\mathrm{d}y, \tag{11}$$

the functions H_ν and z_m were defined above, and

$$\|Q_l\|_{L^2(R^3,1+|x|^a)}\leqslant c\omega^{-l} \qquad \omega\geqslant 1 \qquad a>0,$$

where c is a positive constant which does not depend on ω.

Let us discuss briefly a method for finding $p(x)$ from the measurement of the quantity $\varphi(\zeta,y^1)+\varphi(x^1,\zeta)=l(y_1,x_1)$. Here $y^1=(y_1,b,0)$, $x^1=(x_1,b,0)$, $y_1<x_1$, $b>0$ is fixed, ζ is the point of reflection of the light ray γ_p corresponding to the operator $\nabla^2+\omega^2(1+p(x))$ issued from y^1 and passing through x^1,

$$l(y_1,x_1)=\int_{\gamma_p}(1+p(\eta))^{1/2}\,\mathrm{d}\sigma \tag{12}$$

where $\mathrm{d}\sigma$ is the element of the length of γ_p. Given $l(y_1,x_1)$ we wish to find p. The quantity $l(y_1,x_1)$ is the travel time (up to a constant factor which depends on the units chosen).

Since γ_p depends on $p(x)$, let us linearise the inverse problem by assuming first that γ_p consists of the straight line issued from y^1 with the direction $(1,0,-1)$ reflected at ζ and passing through x^1 with the direction $(1,0,1)$. This path let us denote γ', and let

$$l'(y_1,x_1)=\int_{\gamma'}(1+p(\eta))^{1/2}\,\mathrm{d}\sigma'. \tag{13}$$

Given $l'(y_1,x_1)$ one finds p easily.

Lemma 1. We have

$$(1+p(\zeta))^{1/2}-1=-2^{1/2}\int_0^{\zeta_3}l'_{y_1x_1}(\zeta_1-s,\zeta_1+s,\zeta_2)\,\mathrm{d}s \tag{14}$$

where

$$l'_{y_1x_1}=\frac{\partial^2 l'(y_1,x_1,b)}{\partial y_1\partial x_1}.$$

We use lemma 1 in the iterative inversion procedure which holds for the original (nonlinear) problem (12). Let S be the mapping assigning to $(1+p)^{1/2}$ the function $l(y_1, x_1)$ given by (12), and T be the mapping assigning to $(1+p)^{1/2}$ the function l' given by (13). Formula (14) gives the inverse mapping T^{-1}. Consider the iterative process

$$(1+p_{n+1})^{1/2} = (1+p_n)^{1/2} - T^{-1}[S(1+p_n)^{1/2} - l] \tag{15}$$

where the initial approximation p_0 is an arbitrary function in $L^2(B_R)$ with support in B_R and $l = l(x^1, y^1)$ are the data on P. Note that in order to compute $S(1+p_n)^{1/2}$ one has to solve the direct problem for equation (6) with the known coefficient $p_n(x)$. If $p_n \to p_\infty$ then (15) shows that $S(1+p_\infty)^{1/2} = l$. Thus $p_\infty = p$, where p is the desired coefficient which produces the data f on P. This follows from the injectivity of the mapping $p(x) \to l(x^1, y^1)$.

Convergence of the process (15) is difficult to prove in the three-dimensional case. However, under the assumptions: $p = p(x^3)$, $p \in C^2$, $1 + p(x) > 0$, $p(x_3) = 0$ if $x_3 < -R$, where $R > 0$ is a certain number, one can prove that the iterative process (15) is well defined and converges at the rate of a geometrical series.

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