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LETTER TO THE EDITOR

Offset measurements on a sphere at a fixed frequency do not determine the inhomogeneity uniquely

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Abstract. The integral equation

$$\int_D |x-z|^{-2} v(z)\,\mathrm{d}z = f(x) \qquad |x|=R, \quad D\subset B_R=\{x:|x|\leqslant R\}$$

may have many solutions.

Let $(\nabla^2+k^2 n(x))u=-\delta(x-y)$, $x\in R^3$, $k>0$, $n(x)=1+v(x)$, $v(x)=0$ if $x\notin D\subset R^3$, $v(x)\in L^2(D)$, $D\subset B_R=\{x:|x|<R\}$. Here $R>0$ is an arbitrary large number, u is the acoustic field generated by a source situated at the point y, $v(x)$ is the inhomogeneity and k is the wavenumber. The field u satisfies the equation

$$u(x,y,k)=g(x,y,k)+k^2\int g(x,z,k)v(z)u(z,y,k)\,\mathrm{d}z \tag{1}$$

$$g=\frac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|}. \tag{2}$$

The field $u-g=u_s$ is the scattered field. If $x=y$ then the measurements of $u_s(x,x,k)$ are called the offset measurements. It is easy to see [1] that

$$16\pi^2\lim_{k\to 0} k^{-2}u_s=\int_D\frac{v(z)\,\mathrm{d}z}{|x-z|^2}. \tag{3}$$

Let us denote the left-hand side of (3) by $f(x)$. This is the datum which can be measured. Equation (3) takes the form

$$\int_D |x-z|^{-2}v(z)\,\mathrm{d}z=f(x) \qquad x\in S_R \tag{4}$$

where we assume that the data are measured on the sphere $S_R=\{x:|x|=R\}$.

Problem. Do the data (4) determine the inhomogeneity uniquely?

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It was proved in reference [1] that the equation

$$\int_D (|x-z|\,|y-z|)^{-1}v(z)\,\mathrm{d}z=f(x,y) \qquad x,y\in S_R \tag{5}$$

has at most one solution. This means that the data collected for all positions of the source and receiver on S_R determine the inhomogeneity uniquely.

Result. The data (4) do not determine $v(z)$ uniquely.

Remark. The result holds even if the data are given on any finite number N of spheres S_{R_n}, $R_1<R_2<\ldots<R_N$.

Proof. Since (4) is a linear equation, it is sufficient to construct a non-trivial solution to the equation

$$\int_D |x-z|^{-2}v(z)\,\mathrm{d}z=0 \qquad |x|=R. \tag{6}$$

We construct infinitely many non-trivial solutions to (6). Let us assume that D is a ball, $D=\{z:|z|<a,\ a<R\}$ and $v(z)=v(r)$, $r=|z|$. Then

$$\begin{aligned}\int_{|z|<a} & |x-z|^{-2}v(z)\,\mathrm{d}z\\ &=\int_0^a \mathrm{d}r\,r^2v(r)\int_0^\pi\int_0^{2\pi}\frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{|x|^2+r^2-2r|x|\cos\theta}\\ &=2\pi\int_0^a \mathrm{d}r\,r^2v(r)\int_{-1}^1\frac{\mathrm{d}s}{|x|^2+r^2-2r|x|s}=\frac{\pi}{|x|}\int_0^a \mathrm{d}r\,v(r)r\ln\left(\frac{|x|+r}{|x|-r}\right).\end{aligned} \tag{7}$$

From (6) and (7) it follows that

$$0=\int_{|z|<a}|x-z|^{-2}v(z)\,\mathrm{d}z=\frac{\pi}{R}\int_0^a \mathrm{d}r\,v(r)r\ln\left(\frac{R+r}{R-r}\right). \tag{8}$$

It is clear from (8) that, for any given $R>a$, there exist infinitely many solutions to equation (6): any function $v(r)$ which is orthogonal to $r\ln[(R+r)/(R-r)]$ in $L^2(0,a)$ solves (6). The argument proves the Remark as well.

If D is an arbitrary finite domain then our argument still provides infinitely many non-trivial solutions to (6): just take a ball $B_a\subset D$, set $v(z)=0$ if $z\notin B_a$, define $v(z)=v(|z|)$ in B_a and argue as above.

If the data f are given on the infinite number of spheres S_{R_m} then $v(z)$ is uniquely determined, in particular the function $v(r)$ in (8) is uniquely determined, since the system $\{\ln[(R_n+r)/(R_n-r)]\}$, $n=1,2,\ldots$, $R_n\neq R_m$ if $n\neq m$, $R_n\geqslant R>a$, is complete in $L^2(0,a)$. Indeed, if

$$\int_0^a \mathrm{d}r\,v(r)\ln\left(\frac{R_n+r}{R_n-r}\right)=0 \qquad n=1,2,\ldots,$$

then

$$\sum_{q=0}^{\infty}\int_0^a \mathrm{d}r\,v(r)r^{2q+1}\frac{2}{(2q+1)R_n^{2q+1}}=0 \qquad n=1,2,\ldots.$$

Let $v_q = \int_0^a \mathrm{d}r\, r^{2q+1} v(r)$. Since the function

$$\sum_{q=0}^{\infty} \frac{v_q}{(2q+1)z^{2q+1}} \equiv f(z)$$

is analytic in the region $|z| > a$ and $f(R_n) = 0$, $n = 1, 2, \ldots$, we conclude that $f(z) \equiv 0$, $|z| > a$. Thus $v_q = 0$, $q = 0, 1, 2, \ldots$. By Müntz's theorem [2], $v(r) = 0$.

The same result holds for the equation

$$\int_D \frac{\exp(\mathrm{i}\omega|x-z|)}{|x-z|^2} v(z)\,\mathrm{d}z = 0 \qquad |x| = R$$

where $\omega > 0$ is an arbitrary fixed number.

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References

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