Scattering by a penetrable body

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An integral equation equivalent to the interface problem is derived. A numerical scheme for its solution is given. Convergence of the scheme is established.

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1. INTRODUCTION

Consider the scattering problem:

$$(\nabla^2 + k_0^2)u = 0 \quad \text{in } \Omega, \quad k_0 > 0, \tag{1}$$

$$(\nabla^2 + k_1^2)u = 0 \quad \text{in } \mathscr{D}, \quad k_1 > 0, \tag{2}$$

$$u_+ = u_- \quad \text{on } \Gamma, \tag{3}$$

$$\rho \frac{\partial u_+}{\partial N} = \frac{\partial u_-}{\partial N} \quad \text{on } \Gamma, \tag{4}$$

$$u = u_0 + v, \tag{5}$$

$$\int_{|s| = R} \left| \frac{\partial v}{\partial r} - ik_0 v \right|^2 ds \to 0, \quad R \to \infty. \tag{6}$$

Here u_0 is the incident field which satisfies Eq. (1) in the whole space $\mathbb{R}^3$, $\mathscr{D}$ is a bounded obstacle with a smooth surface Γ, $\mathscr{D}$ is the interior domain, N is the exterior unit normal on Γ pointing into Ω, the exterior domain, k_0 (k_1) is the wave number in Ω ($\mathscr{D}$), $\rho = \text{const} > 0, \rho \neq 1$, the sign $+$ ($-$) denotes the limit value on Γ from the interior (exterior) domain

$$\frac{\partial u_+}{\partial N} \equiv \left(\frac{\partial u}{\partial N}\right)_+ .$$

There is an extensive literature on the exterior boundary value problem. The integral equation method is usually the tool of the studies.[1] The equivalence problem is important in these studies. For the Dirichlet and Neumann boundary conditions (corresponding to scattering by acoustically soft and hard obstacles), the integral equations obtained are not equivalent to the boundary value problems when k_0^2 belongs to some discrete set (the spectrum of the corresponding interior problems). Various ways to modify the integral equations at these exceptional values of k_0^2 were suggested by many authors. We will not discuss this question here since for the problem (1)–(6), the equivalence of the integral equation and the problem (1)–(6) is easy to establish. In Sec. 2 the uniqueness theorem is proved. This theorem is known,[1] but we included a very short proof of it for convenience of the reader. In Sec. 3 existence and uniqueness of the solution to the integral Eq. (17) and the equivalence of this equation to the problem (1)–(6) are proved. In Sec. 4 convergence of a numerical method of solving the integral equation is proved. This result is connected with the T-matrix approach.[2] The integral Eq. (17), which is used in this paper, is of Fredholm's type and is convenient for a numerical treatment. One can derive a boundary integral equation of the type used in the usual T-matrix scheme, but in this equation one has integral operators with strong singularities, and this fact makes the theoretical numerical analysis difficult.

Scattering by a permeable body was discussed recently in Ref. 3, where different integral equations were suggested. Numerical solution of these equations (which involve improper integrals) was not discussed. The basic integral equation in Ref. 3 is of the first kind and its kernel is weakly singular. Thus this equation presents difficulties from the numerical analysis viewpoint.

2. UNIQUENESS OF THE SOLUTION

Theorem 1: If $u_0 = 0$, then the only solution to problem (1)–(6) is $u \equiv 0$.

Proof: If $u_0 = 0$, then u satisfies the radiation condition (6) and $\bar{u}$ solves (1)–(4). Here and below the bar denotes complex conjugation. From Green's formula, it follows that

$$0 = \lim_{R \to \infty} \int_{|s| = R} \left(u \frac{\partial \bar{u}}{\partial r} - \bar{u} \frac{\partial u}{\partial r} \right) ds - \int_\Gamma \left(u \frac{\partial \bar{u}_-}{\partial N} - \bar{u} \frac{\partial u_-}{\partial N} \right) ds. \tag{7}$$

Applying (4) and Green's formula again, one obtains

$$\int_\Gamma \left(u \frac{\partial \bar{u}_-}{\partial N} - \bar{u} \frac{\partial u_-}{\partial N} \right) ds = \rho \int_\Gamma \left(u \frac{\partial \bar{u}_+}{\partial N} - \bar{u} \frac{\partial u_+}{\partial N} \right) ds = 0. \tag{8}$$

From Eq. (7) and (8) it follows that

$$\lim_{R \to \infty} \int_{|s| = R} \left(u \frac{\partial \bar{u}}{\partial r} - \bar{u} \frac{\partial u}{\partial r} \right) ds = 0. \tag{9}$$

Condition (6) for u and (9) yield

$$\lim_{R \to \infty} \int_{|s| = R} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + k_0^2 |u|^2 \right\} ds = 0. \tag{10}$$

From (1) and (10) it follows that $u \equiv 0$. The last conclusion is a well-known result. A short proof of this result can be found in Refs. 4 and 5.

3. BASIC INTEGRAL EQUATION, ITS EQUIVALENCE TO THE PROBLEM (1)–(6). EXISTENCE AND UNIQUENESS OF ITS SOLUTION

Let us rewrite (2) as

$$(\nabla^2 + k_0^2)u = \kappa u, \quad \kappa \equiv k_0^2 - k_1^2. \tag{2'}$$

From the Green's formula, it follows that

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$$\int \{g(\nabla^2 + k_0^2)u + u\delta(x-y)\}dy$$

$$= \kappa \int_{\mathscr{D}} gu\, dy + u(x)$$

$$= \lim_{R\to\infty} \int_{|s|=R} \left(g\frac{\partial u}{\partial r} - u\frac{\partial g}{\partial r}\right)ds - \int_{\Gamma}\left(g\frac{\partial u_-}{\partial N} - u_-\frac{\partial g}{\partial N}\right)ds + \int_{\Gamma}\left(g\frac{\partial u_+}{\partial N} - u_+\frac{\partial g}{\partial N}\right)ds$$

$$= u_0(x) + \int_{\Gamma} g\left(\frac{\partial u_+}{\partial N} - \frac{\partial u_-}{\partial N}\right)ds$$

$$= u_0(x) + (1-\rho)\int_{\Gamma} g\frac{\partial u_+}{\partial N}\, ds, \quad x\notin\Gamma,$$

$$g \equiv \frac{\exp(k_0|x-y|)}{4\pi|x-y|}. \tag{11}$$

This can be written as

$$u(x) = u_0(x) - \kappa T u + (1-\rho)Q\sigma, \tag{12}$$

$$\sigma \equiv \frac{\partial u_+}{\partial N}, \tag{13}$$

$$Tu = \int_{\mathscr{D}} gu\, dy, \quad Q\sigma = \int_{\Gamma} g\sigma\, ds. \tag{14}$$

For any σ, any function u which solves (12) solves (1), (2), (3), (5), and (6). This function will solve (4) iff

$$0 = (\rho-1)\frac{\partial u_0}{\partial N} - \kappa(\rho-1)\frac{\partial Tu}{\partial N} + (1-\rho)\left(\rho\frac{A\sigma+\sigma}{2} - \frac{A\sigma-\sigma}{2}\right)$$

or, which is the same,

$$\sigma = -\frac{2\kappa}{\rho+1}\frac{\partial Tu}{\partial N} + \frac{1-\rho}{1+\rho}A\sigma + \frac{2}{\rho+1}\frac{\partial u_0}{\partial N}, \tag{15}$$

where

$$A\sigma = 2\int_{\Gamma}\frac{\partial g(s,s')}{\partial N_s}\sigma(s')ds',$$

and the known formulas were used:

$$\left(\frac{\partial Q\sigma}{\partial N}\right)_{\pm} = \frac{A\sigma \pm \sigma}{2}. \tag{16}$$

It is easy to check that (15) is equivalent to (13) if one takes as $u(x)$ in (13) the right-hand side of (12). Equations (12) and (15) can be written as

$$w = Bw + h, \tag{17}$$

where

$$w = \begin{pmatrix} u \\ \sigma \end{pmatrix}, \quad B = \begin{pmatrix} -\kappa T & (1-\rho)Q \\ -\dfrac{2\kappa}{\rho+1}\dfrac{\partial}{\partial N}T & \dfrac{1-\rho}{1+\rho}A \end{pmatrix},$$

$$h = \begin{pmatrix} u_0 \\ \dfrac{2}{\rho+1}\dfrac{\partial u_0}{\partial N} \end{pmatrix}. \tag{18}$$

Equation (17) is equivalent to the equation

$$u = u_0 - \kappa Tu + (1-\rho)Q\frac{\partial u_+}{\partial N}. \tag{19}$$

Let us consider B as an operator from $H^q = H_q \oplus \widetilde{H}_{q-1/2}$ into H^q. Here $H_q = W_2^q(\mathscr{D})$, $q\geqslant 0$ is the Sobolev space of functions which are q times differentiable and their derivatives belong to $L^2(\mathscr{D})$, $\widetilde{H}_q = W_2^q(\Gamma)$. It is known[6] that the imbedding $H_{q'}\rightarrow\widetilde{H}_q$ is continuous if $q'\geqslant q+\frac{1}{2}$ and compact if $q'>q+\frac{1}{2}$. For $q<0$ the space H_q is dual to the space $H_{|q|}$ with respect to H_0. Symbol $\oplus$ means that any element $w\in H^q$ is uniquely representable as an ordered pair $\binom{u}{\sigma}$, where $u\in H_q$, $\sigma\in\widetilde{H}_{q-1/2}$, and the inner product in H^q is defined as $(w_1,w_2) = (u_1,u_2)_{H_q} + (\sigma_1,\sigma_2)_{\widetilde{H}_{q-1/2}}$.

Lemma 1: The operator $B{:}H^q\rightarrow H^q$ is compact.

Proof: This follows from the relations: $T{:}H_q\rightarrow H_{q+2}$ is continuous, $Q{:}\widetilde{H}_q\rightarrow H_{q+3/2}$ is continuous, $(\partial/\partial N)T{:}H_q\rightarrow H_{q+1}$ is continuous, $A{:}\widetilde{H}_q\rightarrow\widetilde{H}_{q+1}$ is continuous, and from the compactness of the embeddings: $H_{q'}\rightarrow H_q$ if $q'>q$, $H_{q'}\rightarrow\widetilde{H}_q$ if $q'>q+\frac{1}{2}$.

Lemma 2: Equation (17) and problem (1)–(6) are equivalent.

Proof: 1). (1)–(6)⟹(17). This was shown above in the process of deriving Eq. (17). 2). (17)⟹(1)–(6). If w satisfies (17), then (12) and (15) are satisfied. If u satisfies (12), then u solves (1), (2), (3), (5), and (6). If (15) holds, then $\sigma = \partial u_+/\partial N$ and (4) holds.

Lemma 3: Equation (17) has no more than one solution.

Proof: Equation (17) is equivalent to (1)–(6), and (1)–(6) has no more than one solution (by Theorem 1).

Theorem 2. If $h\in H^q$, then Eq. (1) has a solution in H^q and this solution is unique.

Proof: Theorem 2 follows from Lemmas 1 and 3, Fredholm's alternative, and the inclusion $h\in H^q$.

4. NUMERICAL SOLUTION

Since B is compact in H^0, the convergence of the projection method of solving Eq. (17) is easy to establish (Ref. 4, p. 192). Let us describe the projection method for Eq. (17). Let $\{\phi_j\}$ be a complete linearly independent system of functions in H_0, and $\{\psi_j\}$ be a similar system in $\widetilde{H}_{-1/2}$. The union of the systems

$$\begin{Bmatrix}\phi_j \\ 0\end{Bmatrix}, \begin{Bmatrix}0 \\ \psi_j\end{Bmatrix}$$

is a complete linearly independent system in H^0. Let us take

$$\psi_j = \left(\frac{\partial\phi_j}{\partial N}\right)_+.$$

As $\{\phi_j\}$, let us take the orthonormal system of eigenfunctions of the Dirichlet Laplacian in domain Δ, $\mathscr{D}\subset\Delta$. As Δ one can take, e.g., box or a ball, so that $\{\phi_j\}$ is given explicitly. The system $\{\phi_j\}$ is complete in H_0. The system $\{\partial\phi_j/\partial N\}$ is complete in $\widetilde{H}_{-1/2}$. Indeed, let

$$(*)\int_{\Gamma} f\frac{\overline{\partial\phi_j}}{\partial N}ds = 0, \quad \forall j.$$

The bar denotes complex conjugation. Let us multiply (*) by $\phi_j(x)/k_j^2$, where $(\nabla^2 + k_j^2)\phi_j = 0$ in Δ, $\phi_j = 0$ on $\partial\Delta$, and sum over j. Since $\Sigma_j\phi_j(x)\,\overline{\phi_j(s)}/k_j^2 = G(x,s)$, $-\nabla^2 G = \delta(x-y)$ in Δ, $G=0$ on $\partial\Delta$, this yields

$$v(x) = \int_\Gamma \frac{\partial G(x,s)}{\partial N_s} f(s) ds = 0, \quad x \in \Delta$$

and from jump relations for the potential of double layer, one sees that $f = 0$. Let

$$w_m = \begin{pmatrix} \sum_{j=1}^{m} c_j^{(m)} \phi_j \\ \sum_{j=1}^{m} d_j^{(m)} \frac{\partial \phi_j}{\partial N} \end{pmatrix}. \tag{20}$$

The projection method consists in finding $c_j^{(m)}$, $d_j^{(m)}$ from the linear system

$$(w_m - Bw_m - h, \eta_j)_{H^0} = 0, \quad 1 \leqslant j \leqslant 2m, \tag{21}$$

where

$$\eta_j = \begin{pmatrix} \phi_j \\ 0 \end{pmatrix}, \quad 1 \leqslant j \leqslant m,$$

$$\eta_j = \begin{pmatrix} 0 \\ \frac{\partial \phi_j}{\partial N} \end{pmatrix}, \quad m + 1 \leqslant j \leqslant 2m. \tag{22}$$

The system (21) is a linear system of $2m$ equations for $2m$ unknowns $c_j^{(m)}, d_j^{(m)}$. From the known results [see Refs. 7 and 8 for general theory and Ref. 2(a) and Ref. 4, p. 192 for the problems similar to (21)], it follows that (21) is uniquely solvable for all sufficiently large m and $w_m \to w$ in H^0, where w solves (17).

We reduce by half the number of the unknowns if we use Eq. (15) [the second equation in the vector Eq. (17)] in the form (13), and set

$$d_j^{(m)} = c_j^{(m)}. \tag{23}$$

In this case, the system (21) takes the form

$$\sum_{j=1}^{m} a_{nj} c_j^{(m)} = u_{0n}, \quad 1 \leqslant n \leqslant m, \tag{24}$$

where

$$a_{nj} = (\phi_j, \phi_n) + \kappa (T\phi_j, \phi_n) - (1 - \rho)(Q \frac{\partial \phi_j}{\partial N}, \phi_n), \tag{25}$$

$$u_{0n} = (u_0, \phi_n), \quad (f,g) \equiv (f,g)_{H_0}. \tag{26}$$

The system (24) one can also obtain by applying the projection method to Eq. (19). In this case, the proof of the convergence of the projection scheme requires further study. The reason is that the operator $Q(\partial/\partial N)$ is not compact in H^q.

Remark: If $\rho = 1$ [see (4)], then (19) becomes $u = u_0 - \kappa T u$ and the numerical scheme (24) [with the matrix a_{nj} defined by (25) with $\rho = 1$)] converges in H_q provided that $u_0 \in H_q$. If $\rho = 1$, then it follows from (12) that the values of u in $\mathscr{D}$ define u in the whole space. Therefore, if $u_m(x)$ is the approximate solution defined from (24) by the formula $u_m \equiv \Sigma_{j=4}^{m} c_j^{(m)} \phi_j$, then the function $u_0(x) - \kappa T u_m$ converges to the solution $u(x)$ of the problem (1)–(6) with $\rho = 1$ in H_q, in $\mathscr{C}(\Omega_R)$, where $\Omega_R \equiv \{x: |x| > R\}$ and $\mathscr{D} \subset \mathscr{B}_R$, $\mathscr{B}_R \equiv \{x: |x| \leqslant R\}$. If $q \geqslant 2$, then $u_0(x) - \kappa T u_m \to u(x)$ in $\mathscr{C}(\mathbb{R}^3)$ as $m \to \infty$.

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