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In: Acoustic, Electromagnetic and Elastic wave scattering – Focus on the T-matrix approach.

Pergamon Press, New York, 1980, pp. 537-546. (Ed. V. Varadan and V. Varadan).

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1 Introduction

The theory of wave scattering by small bodies was initiated by Rayleigh (1871). Thompson (1893) was the first to understand the role of magnetic dipole radiation. Since then, many papers have been published on the subject because of its importance in applications. From a theoretical point of view there are two directions of investigation:

(i) to prove that the scattering amplitude can be expanded in powers of ka, where $k = 2\pi/\lambda$ and a is a characteristic dimension of a small body,

(ii) to find the coefficients of the expansion efficiently.

Stevenson (1953), Senior and Kleinman can be mentioned among contributors to the first topic.

To my knowledge, there were no results concerning the second topic for bodies of an arbitrary shape. Such results are of interest in geophysics, radiophysics, optics, colloidal chemistry and solid state theory.

In this paper we review the results of the author on the theory of scalar and vector wave scattering by small bodies of an arbitrary shape with the emphasis on practical applicability of the formulas obtained and on the mathematical rigor of the theory. For the scalar wave scattering by a single body, our main results can be described as follows:

(1) Analytical formulas for the scattering amplitude for a small body of an arbitrary shape are obtained; dependence of the scattering amplitude on the boundary conditions is described.

*key words: wave scattering, small bodies of arbitrary shapes
†Math subject classification: 65J10, 65R20, 78A45

(2) An analytical formula for the scattering matrix for electromagnetic wave scattering by a small body of an arbitrary shape is given. Applications of these results are outlined (calculation of the properties of a rarefied medium; inverse radio measurement problem; formulas for the polarization tensors and capacitance).

(3) The multi-particle scattering problem is analyzed and interaction of the scattered waves is taken into account. For the self-consistent field in a medium consisting of many particles ($\sim 10^{23}$), integral-differential equations are found. The equations depend on the boundary conditions on the particle surfaces. These equations offer a possibility of solving the inverse problem of finding the medium properties from the scattering data. For about 5 to 10 bodies the fundamental integral equations of the theory can be solved numerically to study the interaction between the bodies.

In section 2 the results concerning the scalar wave scattering are described. In section 3 the electromagnetic scattering is studied and the solution of the inverse problem of radio measurements is outlined. In section 4 the many-body problem is examined.

2 Scalar Wave Scattering by a Single Body

Consider the problem

$$\left(\nabla^2 + k^2\right) v = 0 \text{ in } \Omega; \quad \left(\frac{\partial v}{\partial N} - hv\right)\Big|_\Gamma = -\left(\frac{\partial u_0}{\partial N} - hu_0\right)\Big|_\Gamma, \tag{1}$$

$$v \sim \frac{\exp\left(ik|x|\right)}{|x|} f(n,k) \text{ as } |x| \to \infty, \quad \frac{x}{|x|} = n, \tag{2}$$

where $\Omega = R^3 \backslash D$, D is a bounded domain with a smooth boundary Γ, N is the outer normal to Γ, u_0 is the initial field which is usually taken in the form $u_0 = \exp\{ik(\nu, x)\}$. We look for a solution of the problem (1)-(2) of the form

$$v = \int_\Gamma \frac{\exp\left(ikr_{xt}\right)\sigma(t)}{4\pi r_{xt}} dt, \quad r_{xt} = |x - t|, \tag{3}$$

and for the scattering amplitude f we have the formula

$$f = \frac{1}{4\pi}\int_\Gamma \exp\{-ik(n,t)\}\sigma(t,k)dt = \frac{1}{4\pi}\int_\Gamma \sigma_0(t)dt + O(ka), \tag{4}$$

where

$$\sigma(t,k) = \sigma_0(t) + ik\sigma_1(t) + \frac{(ik)^2}{2}\sigma_2(t) + \dots \tag{5}$$

Putting (3) in the boundary conditions (1) we get the integral equation for σ:

$$\sigma = A(k)\sigma - hT(k)\sigma - 2hu_0 + 2\frac{\partial u_0}{\partial N}, \tag{6}$$

where

$$A(k)\sigma = \int_\Gamma \frac{\partial}{\partial N_s} \quad \frac{\exp(ikr_{st})}{2\pi r_{st}}\sigma(t)dt, \quad T(k)\sigma = \int_\Gamma \frac{\exp(ikr_{st})}{2\pi r_{st}}\sigma(t)dt. \tag{7}$$

Expanding σ, $A(k)$ and $T(k)$ in the powers k and equating the corresponding terms in (6) we obtain, for $h = 0$, i.e. for the Neumann boundary condition, the following equations:

$$\sigma_0 = A_0\sigma_0, \tag{8}$$

$$\sigma_1 = A_0\sigma_1 + A_1\sigma_0 + 2\frac{\partial u_{01}}{\partial N}, \tag{9}$$

$$\sigma_2 = A_0\sigma_2 + 2A_1\sigma_1 + A_2\sigma_0 + 2\frac{\partial u_{02}}{\partial N}, \tag{10}$$

etc, where

$$A(k) = A_0 + ikA_1 + \frac{(ik)^2}{2}A_2 + \dots; \quad u_0 = u_{00} + iku_{01} + \frac{(ik)^2}{2}u_{02} + \dots. \tag{11}$$

Expanding f in formula (4) we obtain, up to the terms of the second order:

$$\begin{aligned} f \quad &= \tfrac{1}{4\pi}\int_\Gamma \sigma_0 dt + \tfrac{ik}{4\pi}\left\{\int_\Gamma \sigma_1 dt - \left(n, \int_\Gamma \sigma_0(t)tdt\right)\right\} \\ &+ \tfrac{(ik)^2}{8\pi}\left\{\int_\Gamma \sigma_2 dt - 2\left(n, \int_\Gamma \sigma_1 tdt\right) + \int_\Gamma \sigma_0(n,t)^2 dt\right\}. \end{aligned} \tag{12}$$

From (8) it follows that $\sigma_0 = 0$ and from (9) it follows that $\int_\Gamma \sigma_1 dt = 0$. Some calculations lead to the following final result (see [17]):

$$f = \frac{ikV}{4\pi}\beta_{pq}n_p\frac{\partial u_0}{\partial x_q}\bigg|_{x=0} + \frac{V}{4\pi}\Delta u_0\bigg|_{x=0}. \tag{13}$$

Usually $u_0 = \exp\{ik(\nu, x)\}$, and in this case formula (13) can be written as:

$$f = -\frac{k^2V}{4\pi}\left(\beta_{pq}\nu_q n_p + 1\right), \tag{13'}$$

where over the repeated indices the summation is understood, V is the volume of the body D and β_{pq} is the magnetic polarizability tensor of D. Note that $f \sim k^2a^3$, the scattering is anisotropic and is defined by the tensor β_{pq}. Formula (25) below allows one to calculate β_{pq}.

For $h = \infty$ (the Dirichet boundary condition) integral equation (6) takes the form:

$$T(k)\sigma = -2u_0. \tag{14}$$

Hence

$$\int_\Gamma \frac{\sigma_0 dt}{4\pi r_{st}} = -u_0\bigg|_\Gamma.$$

Since $ka << 1$ the field $u_0\left|_\Gamma = u_0(x,k)\right|_{x=0}$, where the origin is assumed to be inside the body D. From the above equation it follows that $\int_\Gamma \sigma_0 dt = -Cu_0$, and

$$f = -\frac{Cu_0}{4\pi}, \tag{15}$$

where C is the capacitance of a conductor with the shape D. Hence, for the Dirichet boundary condition, $f \sim a$, where a is a characteristic length of D, and the scattering is isotropic.

For $h \neq 0$, using the same line of arguments, it is possible to obtain the following approximate formula for the scattering amplitude:

$$f \approx -\frac{hS}{4\pi\left(1+hSC^{-1}\right)} \quad u_{00}, \tag{16}$$

where $S =$ meas (Γ), i.e., the surface area of Γ, and C is the capacitance of D. If h is very small $\left(h \sim k^2a^3\right)$, then the formula for f should be changed and the terms analogous to (13) should be taken into account.

3 Electromagnetic Wave Scattering by a Single Body

If a homogeneous body D with the parameters ε, μ, σ, is placed into a homogeneous medium with the parameters ε_0, μ_0, σ_0, then the following formula for the scattering matrix $\mathcal{S}$ was established by the author (see [17]):

$$\mathcal{S} = \frac{k^2V}{4\pi}\left[\begin{array}{ll} \mu_0\beta_{11} + \alpha_{22}\cos\theta - \alpha_{32}\sin\theta, & \alpha_{21}\cos\theta - \alpha_{31}\sin\theta - \mu_0\beta_{12} \\ \alpha_{12} - \mu_0\beta_{21}\cos\theta + \mu_0\beta_{31}\sin\theta, & \alpha_{11} + \mu_0\beta_{22}\cos\theta - \mu_0\beta_{32}\sin\theta \end{array}\right], \tag{17}$$

where $\mathcal{S}$ is defined by the formula

$$\left(\begin{array}{c} f_2 \\ f_1 \end{array}\right) = \mathcal{S}\left(\begin{array}{c} E_2 \\ E_1 \end{array}\right) = \left(\begin{array}{c} S_2S_3 \\ S_4S_1 \end{array}\right)\left(\begin{array}{c} E_2 \\ E_1 \end{array}\right), \tag{18}$$

θ is the angle of scattering, E_1, E_2 are the components of the initial field, f_1, f_2 are the components of the scattered field in the far field region multiplied by $|x|^{-1}\exp(ik|x|)$, the plane YOZ is the plane of scattering, $\alpha_{ij} = \alpha_{ij}(\gamma)$, is the polarizability tensor, $\gamma = (\varepsilon - \varepsilon_0)\,/\,(\varepsilon + \varepsilon_0)$, and $\beta_{ij} = \alpha_{ij}(-1)$ is the magnetic polarizability tensor.

If one knows $\mathcal{S}$ one can find all the values of interest to the physicists for electromagnetic wave propagation in a rarefied medium consisting of small bodies. The tensor of refraction coefficient can be calculated by the formula $n_{ij} = \delta_{ij} + 2\pi Nk^{-2}S_{ij}(0)$, where N is the number of bodies per unit volume, and $S_{ij}(0)$ are the elements of the S-matrix corresponding to the forward scattering, that is for $\theta = 0$. The tensor $\alpha_{ij}(\gamma)$ can be calculated analytically by the formula (see [17]):

$$\left|\alpha_{ij}(\gamma) - \alpha_{ij}^{(n)}(\gamma)\right| \leq Aq^n, \quad 0 < q < 1, \quad \gamma := \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0}, \tag{19}$$

where A, and q are some constants depending only on the geometry of the surface, and

$$\alpha_{ij}^{(n)}(\gamma) := \frac{2}{V} \sum_{m=0}^{n} \frac{(-1)^m}{(2\pi)^m} \quad \frac{\gamma^{n+2} - \gamma^{m+1}}{\gamma - 1} b_{ij}^{(m)}, \quad n \geq 1. \tag{20}$$

In (20)

$$b_{ij}^{(0)} = V\delta_{ij}, \quad b_{ij}^{(1)} = \int_\Gamma \int_\Gamma \frac{N_i(s)N_j(t)dsdt}{r_{st}}, \tag{21}$$

$$b_{ij}^{(m)} = \int_\Gamma \int_\Gamma dsdt N_i(s)N_j(t) \underbrace{\int_\Gamma \dots \int_\Gamma}_{m-1} \tfrac{1}{r_{st}} \psi\left(t_1, t\right) \dots \psi\left(t_{m-1}, t_{m-2}\right) dt_1 \dots dt_{m-1},$$
$$\psi(t, s) \equiv \tfrac{\partial}{\partial N_t} \quad \tfrac{1}{r_{st}}. \tag{22}$$

In particular

$$\alpha_{ij}^{(1)}(\gamma) = 2\left(\gamma + \gamma^2\right)\delta_{ij} - \frac{\gamma^2 b_{ij}^{(1)}}{\pi V}, \quad \beta_{ij}^{(1)} = -\frac{b_{ij}^{(1)}}{\pi V}. \tag{23}$$

For the particles with $\mu = \mu_0$ and ε not very large, so that the depth δ of the skin layer is considerably larger than a, one can neglect the magnetic dipole radiation and in formula (17) for the scattering matrix one can omit the terms with the multipliers β_{ij}.

The vectors of the electric P and magnetic M polarizations can be found by the following formulas, respectively,

$$P_i = \alpha_{ij}(\gamma)V\varepsilon_0 E_j, \quad \gamma := \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0}, \tag{24}$$

where E_j is the initial field, over the repeated indices one sums up, and

$$M_i = \alpha_{ij}(\widetilde{\gamma})V\mu_0 H_j + \beta_{ij}V\mu_0 H_j, \quad \widetilde{\gamma} := \frac{\mu - \mu_0}{\mu + \mu_0}, \quad \beta_{ij} := \alpha_{ij}(-1), \tag{25}$$

where H_j is the initial field, and the second term on the right hand side of equality (25) should be omitted if the skin-layer depth $\delta >> a$.

The scattering amplitudes can be found from the formulas:

$$f_E = \frac{k^2}{4\pi\varepsilon_0}[n, [P, n]] + \frac{k^2}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}[M, n], \tag{26}$$

$$f_H = \sqrt{\frac{\varepsilon_0}{\mu_0}}[n, f_E], \tag{27}$$

where $[A, B]$ stands for the vector product $A \times B$, and P and M can be calculated by formulae (24), (25), (19), (20), (22). If $\delta >> a$ one can neglect the second term on the right-hand side of (26).

It is possible to give a simple solution to the following inverse problem which can be called *the inverse problem of radiomeasurements.*

Suppose an initial electromagnetic field is scattered by a small probe. Assume that the scattered field E', H' can be measured in the far field region. The problem is: calculate the initial field at the point where the small probe detects E', H'. This problem is of interest, for example, when one wants to determine the electromagnetic field distribution in an antenna's aperture. Let us assume for simplicity that for the probe $\delta >> a$, so that

$$E' = \frac{\exp(ikr)}{r} \frac{k^2}{4\pi\varepsilon_0} [n[P, n]] \,. \tag{28}$$

From (28) one can find $P - n_1 (P, n_1) = E' (n_1) b$ where $b = \frac{\exp(ikr)}{r} \frac{k^2}{4\pi\varepsilon_0}$. A measurement in the n_2 direction, where $(n_1, n_2) = 0$, results in $P - n_2(P, n_2) = E'(n_2)b$. Hence $(n_1, P) = b(E'(n_2), n_1)$. Thus $P = b\{E'(n_1) + n_1(E'(n_2), n_1)\}$. But (cf (24)):

$$P_i = \alpha_{ij}(\gamma) V \varepsilon_0 E_j, \quad 1 \le i \le 3. \tag{$*$}$$

Since V and ε_0 are known and $\alpha_{ij}(\gamma)$ can be calculated by formulae (19), (20) and the matrix α_{ij} is positive definite (because $\frac{1}{2}\alpha_{ij} V \varepsilon_0 E_j E_i$ is the energy) it follows that system $(*)$ is uniquely solvable for E_j. Its solution is the desired vector E.

Let us give a formula for the capacitance of a conductor D of an arbitrary shape, which proved to be very useful in practice (see [17]):

$$C^{(n)} = 4\pi\varepsilon_0 S^2 \left\{ \frac{(-1)^n}{(2\pi)^n} \int_\Gamma \int_\Gamma \frac{dsdt}{r_{st}} \underbrace{\int_\Gamma \dots \int_\Gamma}_{n \text{ times}} \psi(t, t_1) \dots \psi(t_{n-1}, t_n) dt_1 \dots dt_n \right\}^{-1}, \tag{29}$$

$$C^{(0)} = \frac{4\pi\varepsilon_0 S^2}{J} \le C, \quad J \equiv \int_\Gamma \int_\Gamma \frac{dsdt}{r_{st}}, \quad S = \text{meas}\Gamma. \tag{30}$$

It can be proved that

$$\left| C - C^{(n)} \right| \le A q^n, \quad 0 < q < 1, \tag{30'}$$

where A and q are constants which depend only on the geometry of Γ.

Remark 3.1. *The theory is also applicable for small layered bodies (see [17]).*

Remark 3.2. *Two sided variational estimates for α_{ij} and C were given in [17] and [18].*

4 Many Body Wave Scattering

First we describe a method for solving the scattering problem for r bodies, $r \sim 5-10$, and then we derive an integral-differential equation for the self-consistent field in a medium consisting of many ($r \sim 10^{23}$) small bodies. We look for a solution of the scalar wave scattering problem of the form

$$u = u_0 + \sum_{j=1}^{r} \int_{\Gamma_j} \frac{\exp(ikr_{st})}{4\pi r_{xt}} \sigma_j(t)dt. \tag{31}$$

Applying the boundary condition,

$$u\Big|_{\Gamma_j} = 0, \quad 1 \leq j \leq r, \tag{32}$$

we obtain the system of r integral equations for the r unknown functions σ_j. In general this system can be solved numerically. When $d << \lambda$, where $d = \min_{i \neq j} d_{ij}$, and d_{ij} is the distance between $i - th$ and $j - th$ body, the system of the integral equations has dominant diagonal terms and it can be easily solved by an iterative process, the zero approximation being the initial field u_0.

If $ka >> 1$, $d >> a$, but not necessarily $d >> \lambda$, the average (self-consistent) field in the medium consisting of small particles can be found from the integral equation ([17]):

$$u(x,k) = u_0(x) - \int_{R^3} \frac{\exp(ikr_{xy})}{4\pi r_{xy}} q(y)u(y,k)dy. \tag{33}$$

Here $q(y)$ is the average value of $h_j S_j \left(1 + h_j S_j C_j^{-1}\right)^{-1}$ over the volume dy in a neighborhood of y for bodies with impedance boundary conditions. For $h_j = \infty$ (the Dirichlet boundary condition) and identical bodies, one has $q(y) = N(y)C$, where $N(y)$ is the number of the bodies per unit volume and C is the capacitance of a body. For the Neumann boundary condition the corresponding equation is the integral-differential equation:

$$u(x,k) = u_0(x,k) + \int_{R^3} \frac{\exp(ikr_{xy})}{4\pi r_{xy}} \left\{ B_{pq}(y) \frac{\partial u(y,k)}{\partial y_q} \quad \frac{x_p - y_p}{r_{xy}} + \frac{1}{2} b(y) \Delta u(y,k) \right\} dy, \tag{34}$$

where

$$b(y) = N(y)V, \quad B_{pq}(y) = ikV\beta_{pq}N(y), \tag{35}$$

V is the volume of a body, and β_{pq} is its magnetic polarizability tensor (see forumla (25)). The solution to equations (33) and (34) can be considered as the self-consistent (effective) field acting in the medium.

Equations (33) and (34) allow one to solve the inverse problems of the determination of the medium properties from the scattering data. For example, from (33) it follows that the scattering amplitude has the form

$$f = -\frac{1}{4\pi} \int \exp\{-ik(n,y)\} q(y)u(y,k)dy. \tag{36}$$

For a rarefied medium it is reasonable to replace u by u_0 (the Born approximation) and to obtain

$$f \approx -\frac{1}{4\pi}\int_{R^3} \exp\{-ik(n,y)\}q(y)u_0(y,k)dy. \tag{37}$$

If $u_0 = \exp\{ik(\nu,x)\}$ formula (37) is valid for $k >> 1$ with the error $O(k^{-1})$ if

$$|q(x)| + |\nabla q(x)| \leq \frac{c}{1+|x|^{3+\varepsilon}}, \quad \varepsilon > 0.$$

Hence if f is known for all $0 < k < \infty$, and all $(n-\nu) \in S_2$, where S_2 is the unit sphere in R^3, the Fourier transform of $q(y)$ is known, and q can be uniquely determined. If $q(y)$ is compactly supported, i.e. $q(y) = 0$ outside some bounded domain, then f is an entire function of k, and knowing f in any interval $[k_0, k_1]$, $0 < k_0 < k_1$, for all $(n-\nu) \in S_2$ one can find f for all $0 < k < \infty$ uniquely by analytic continuation, and thus one can determine $q(y)$ uniquely.

Let us consider the r-body problem for a few bodies, $r \sim 10$ (small r). Assume that the Dirichlet boundary condition holds. Let us look for a solution of the form

$$u(x) = u_0 + \sum_{j=1}^{r} \int_{\Gamma_j} \frac{\exp(ik|x-t|)}{4\pi|x-t|}\sigma_j(t)dt. \tag{38}$$

The scattering amplitude is equal to

$$f(n,k) = \frac{1}{4\pi}\sum_{j=1}^{r} \exp\{-ik(n,t_j)\} \int_{\Gamma_j} \exp\{-ik(n,t-t_j)\}\sigma_j(t)dt, \tag{39}$$

where t_j is some point inside the j-th body. Since $ka << 1$ this formula can be rewritten as:

$$f(n,k) = \frac{1}{4\pi}\sum_{j=1}^{r} \exp\{-ik(n,t_j)\} Q_j, \tag{40}$$

where

$$Q_j = \int_{\Gamma_j} \sigma_{j0}dt + O(ka), \quad \sigma_{j0} = \sigma_j\Big|_{k=0}. \tag{41}$$

This is the same line of arguments as in Section 2. Using the boundary condition one gets:

$$\sum_{j\neq m, j=1}^{r} \int_{\Gamma_j} \frac{\exp(ik|x_m-t|)}{4\pi|x_m-t|}\sigma_j(t)dt + \int_{\Gamma_m} \frac{\exp(ik|x_m-t|)}{4\pi|x_m-t|}\sigma_m dt = -u_0(x_m). \tag{42}$$

With the accuracy of $O(ka + \frac{a}{d})$, this can be written as:

$$\int_{\Gamma_m} \frac{\sigma_m(t)dt}{4\pi|x_m-t|} + \sum_{j=1, j\neq m}^{r} \frac{\exp(ikd_{mj})}{4\pi d_{mj}}Q_j = -u_{0m}, \quad 1 \leq m \leq r, \tag{43}$$

where $d_{mj} = |x_m - t_j|$. If C_m is the capacitance of the m-th body one can rewrite (43) as:

$$Q_m = -C_m u_{0m} - \sum_{j=1, j\neq m}^{r} C_m \frac{\exp(ikd_{mj})}{4\pi d_{mj}} Q_j, \quad 1 \leq m \leq r. \tag{44}$$

This is a linear algebraic system from which Q_j, $1 \leq m \leq r$, can be found. If $d_{mj} C_m^{-1} >> 1$, this system can be easily solved by an iterative process. If $\{Q_j\}$ are known, then the scattering amplitude can be found from (40).

More details about the described theory the reader can find in the References, especially in he monograph [17].

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