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Reproducing kernel

Consider an abstract set E and a linear set F of functions $f : E \to \mathbb{C}$.

Assume that F is equipped with an inner product (f, g) and F is complete with respect to the norm $\| f \|= (f, f)^{\frac{1}{2}}$. Then F is a Hilbert space H.

A function $K(x, y)$, $x, y \in E$, is called a reproducing kernel (rk) of H if and only if the following two conditions are satisfied:

i) for every fixed $y \in E$, the function $K(x, y) \in H$

and

ii) $(f(x),\, K(x, y)) = f(y)\ \forall f \in H$.

This definition is given in [1], see also [6].

Properties of the reproducing kernels:

1) if a reproducing kernel $K(x, y)$ exists it is unique,

2) a reproducing kernel $K(x, y)$ exists if and only if $|f(y)| \leq c(y) \| f \| \quad \forall f \in H$,

3) $K(x, y)$ is a nonnegative-definite kernel, that is,

$$\sum_{i,j=1}^{n} K(x_i, x_j) t_j \bar{t}_i \geq 0 \quad \forall x_i, y_j \in E, \quad \forall t \in \mathbb{C}^n,$$

where the overbar stands for complex conjugate.

In particular, property 3) implies:

$$K(x, y) = \overline{K(y, x)}, \quad K(x, x) \geq 0, \quad |K(x, y)|^2 \leq K(x, x) K(y, y).$$

Every nonnegative-definite kernel $K(x, y)$ generates a Hilbert space H_K for which $K(x, y)$ is a reproducing kernel (see also reproducing kernel Hilbert space, RKHS).

If $K(x, y)$ is a rk, then the operator $Kf := (Kf)(\cdot) := (f, K(x, \cdot)) = f(\cdot)$ is injective: $Kf = 0$ implies $f = 0$ by reproducing property ii), and $K : H \to H$ is surjective. Therefore the inverse operator K^{-1} is defined on $R(K) = H$, and since $Kf = f$, the operator K is the identity operator on H_K, and so is its inverse.

Examples of reproducing kernels.

1. Consider a Hilbert space H of analytic in a bounded simply-connected domain D of the complex z-plane. If $f(z)$ is analytic in D, $z_0 \in D$ and the disc $D_{z_0,r} := \{z : |z - z_0| \leq r\} \in D$, then

$$|f(z_0)|^2 \leq \frac{1}{\pi r^2} \int_{D_{z_0,r}} |f(\zeta)|^2 dx dy \leq \frac{1}{\pi r^2} (f, f)_{L^2(D)}.$$

Therefore H is a RKHS. Its rk $K_D(z,\zeta)$ is called Bergman's kernel.

If $\{\phi_j(z)\}$ is an orthonormal basis of H, $\phi_j \in H$, then $K_D(z,\zeta) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(\zeta)}$.

If $w = f(z, z_0)$ is the conformal map of D onto the disc $|w| \leq \rho_D$, such that $f(z, z_0) = 0$, $f'(z_0, z_0) = 1$, then [2]:

$$f(z, z_0) = \frac{1}{K_D(z_0, z_0)} \int_{z_0}^{z} K_D(t, z_0)dt.$$

Let T be a domain in $\mathbb{R}^n$ and $h(t,p) \in L^2(T, dm)$ for every $p \in E$. Here $m(t) > 0$ is a finite measure on T.

Define a linear map $L : L^2(T, dm) \to F$

$$f(p) = Lg := \int_T g(t)\overline{h(t,p)}dm(t) \tag{1}$$

Define the kernel

$$K(p,q) := \int_T h(t,q)\overline{h(t,p)}dm(t) \quad p,q \in E. \tag{2}$$

This kernel is nonnegative-definite:

$$\sum_{i,j+1}^{n} K(p_i,p_j)\xi_j\overline{\xi_i} = \int_T \left|\sum_{j=1}^{n} \xi_j h(t,p_j)\right|^2 dm(t) > 0 \text{ if } \xi \neq 0$$

provided that for any set $\{p_1, \ldots, p_n\} \in E$ the set of functions $\{h(t,p_j)\}_{1\leq j\leq n}$ is linearly independent in $L^2(T, dm)$.

In this case the kernel $K(p,q)$ generates a uniquely determined RKHS H_K for which $K(p,q)$ is the reproducing kernel.

In [6] it is claimed that a convenient characterization of the range $R(L)$ of linear transform (1) is given by the formula $R(L) = H_K$. In [4] it is shown by examples that such a characterization is often useless in practice: the norm in H_K in general cannot be described in terms of the standard Sobolev or Hölder norms, and the assumption in [6] that H_K can be realized as $L^2(E, d\mu)$ is not justified and is not correct, in general.

However, in [6] there are some examples of characterizations of H_K for some special operators L and in [5] a characterization of the range of a wide class of multidimensional linear transforms, whose kernels are kernels of positive rational functions of selfadjoint elliptic operators, is given.

Reproducing kernels are discussed in [5] for the rigged triples of Hilbert spaces. If H_0 is a Hilbert space and $A > 0$ is a linear compact operator defined on all of H, then the closure of H_0 in the norm $(Au,u)^{\frac{1}{2}} =\parallel A^{\frac{1}{2}}u \parallel$ is a Hilbert space $H_- \supset H_0$. The dual space to H_-, with respect to H_0 is denoted by H_+, $H_+ \subset H_0 \subset H_-$. The inner product in H_+ is given by the formula $(u,v)_+ = (A^{-\frac{1}{2}}u, A^{-\frac{1}{2}}v)_0$. The space $H_+ = R(A^{\frac{1}{2}})$, equipped with this inner product, is a Hilbert space.

Let $A\varphi_j = \lambda_j\varphi_j$, where the eigenvalues λ_j are counted according to their multiplicities and $(\varphi_j, \varphi_m)_0 = \delta_{jm}$, where δ_{jm} is the Kronecker delta.

Let us assume that $|\varphi_j(x)| < c$ for all j and all x, and $\Lambda^2 := \sum_{j=1}^\infty \lambda_j < \infty$.

Then H_+ is a RKHS and its reproducing kernel is $K(x,y) = \sum_{j=1}^\infty \lambda_j \varphi_j(y)\overline{\varphi_j(x)}$.

To check that $K(x,y)$ is indeed the reproducing kernel of H_+, one calculates $(A^{-\frac{1}{2}}u, A^{-\frac{1}{2}}K)_0 = (u, A^{-1}K)_0 = u(y)$. Indeed, $A^{-1}K = I$ is the identity operator because $Au = \sum_{j=1}^\infty \lambda_j (u, \varphi_j)\varphi_j(x)$, so that $K(x,y)$ is the kernel of the operator A in H_0.

The value $u(y)$ is a linear functional in H_+, so that H_+ is a RKHS. Indeed, if $u \in H_+$, then $v := A^{-\frac{1}{2}}u \in H_0$. Therefore, denoting $v_j := (v, \varphi_j)_0$ and using the Cauchy inequality and Parseval's equality one gets: $|u(y)| = |\sum_{j=1}^\infty \lambda_j^{\frac{1}{2}} v_j \varphi_j(y)| < c\Lambda ||v||_0 = c\Lambda||u||_+$, as claimed.

From the representation of the inner product in the RKHS H_+ by the formula $(u,v)_+ = (A^{-\frac{1}{2}}u, A^{-\frac{1}{2}}v)_0$ it is clear that, in general, the inner product in H_+ is not an inner product in $L^2(E, d\mu)$.

The inner product in H_+ is of the form

$$(u,v)_+ = \int_D \int_D B(x,y)u(y)\overline{v(x)}dydx \text{ if } H_0 = L^2(D),$$

where the distributional kernel $B(x,y) = \sum_{j=1}^\infty \lambda_j^{-1} \varphi_j(x)\overline{\varphi_j(y)}$ acts on $u \in R(A)$ by the formula $\int_D B(x,y)u(y)dy = \sum_{j=1}^\infty \lambda_j^{-1}(u, \varphi_j)_0 \varphi_j(x)$, where $(u, \varphi_j)_0 := \int_D u(y)\overline{\varphi_j(y)}dy$ is the Fourier coefficient of u. If $u \in R(A)$, then $u = Aw$ for some $w \in H_0$, and $(u, \varphi_j) = \lambda_j w_j$. Thus the series $\sum_{j=1}^\infty \lambda_j^{-1}(u,\varphi_j)_0\varphi_j(x) = \sum_{j=1}^\infty w_j \varphi_j(x) = w(x)$ converges in $H_0 = L^2(D)$.

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