

Encyclopedia of Mathematics, Supplemental Vol. 3, Kluwer Acad. Publishers, Dordrecht, 2001, 310-311

Pseudolocal tomography.

Let $f(x)$ be a piecewise smooth compactly supported function, $x \in \mathbb{R}^2$. The Radon transform $\widehat{f}(\alpha, p) := Rf$ is defined by the formula $\widehat{f}(\alpha, p) = \int_{\ell_{\alpha p}} f(x) ds$, where $\ell_{\alpha p} := \{x : \alpha \cdot x = p\}$ is a straight line parametrized by a unit vector $\alpha \in S^1$, S^1 is the unit circle in \mathbb{R}^2 , and $p \in \mathbb{R}_+ := [0, \infty)$. By definition, $\widehat{f}(-\alpha, -p) = \widehat{f}(\alpha, p)$. By local tomographic data one means the values of $\widehat{f}(\alpha, p)$ for α and p satisfying the condition $|\alpha \cdot x_0 - p| < \rho$, where x_0 is a given point and $\rho > 0$ is a small number. Thus local tomographic data are the line integrals of $f(x)$ for the lines intersecting “the region of interest”, the disc centered at x_0 of radius ρ .

It is not possible, in general, to find $f(x_0)$ from the local tomographic data [1]. What practically useful information about $f(x)$ can one get from these data? Information, very useful practically, is the location of discontinuity curves of $f(x)$ and the sizes of the jumps of $f(x)$ across these curves.

Pseudolocal tomography solves the problem of finding the above information from the local tomographic data.

This is done by computing the pseudolocal tomography function introduced in [1]:

$$f_\rho(x) := \frac{1}{4\pi^2} \int_{S^1} \int_{\alpha \cdot x - \rho}^{\alpha \cdot x + \rho} \frac{\widehat{f}_p(\alpha, p)}{\alpha \cdot x - p} dp d\alpha, \quad (1)$$

where $\widehat{f}_p := \frac{\partial \widehat{f}}{\partial p}$. The known inversion formula reads:

$$f(x) = \frac{1}{4\pi^2} \int_{S^1} \int_{-\infty}^{\infty} \frac{\widehat{f}_\rho(\alpha, p)}{\alpha \cdot x - p} dp d\alpha, \quad (2)$$

so that (1) is based on the following idea: keep a small neighborhood of the singular point $p = \alpha \cdot x$ in (1) and neglect the rest of the Cauchy integral in (1).

By definition one needs only the local tomographic data to calculate the pseudolocal tomography function $f_\rho(x)$.

The basic result is: $f(x) - f_\rho(x) \in C(R^2)$ is a continuous function [1], [2].

Therefore, $f(x)$ and $f_\rho(x)$ have the same discontinuity curves and the same sizes of the jumps across discontinuities.

It is also proved in [1] that if $f \in C^2(U)$, where U is an open set, $U \subset \mathbb{R}^2$, then the function $f_\rho^C(x) := f(x) - f_\rho(x)$ has the following properties:

$$|f_\rho^C(x) - f(x)| = O(\rho) \text{ as } \rho \rightarrow 0, \quad x \in U, \quad (3)$$

and the convergence in (3) is uniform on compact subsets of U . If $x_0 \in S$, where S is a smooth discontinuity curve of $f(x)$, then

$$\left| f_\rho^C(x_0) - \frac{f_+(x_0) + f_-(x_0)}{2} \right| = O(\rho \ln \rho) \text{ as } \rho \rightarrow 0. \quad (4)$$

Here $f_\pm(x_0)$ are the limiting values of $f(x)$ as $x \rightarrow x_0$ from different sides of S along a path nontangential to S .

If n_0 is a unit vector normal to S at the point x_0 , then for an arbitrary $\gamma \in \mathbb{R}$, $\gamma \neq 0$, one has

$$\lim_{\rho \rightarrow 0} [f(x_0 + \gamma \rho n_0) - f_\rho^C(x_0 + \gamma \rho n_0)] = D(x_0)\psi(\gamma),$$

where

$$D(x_0) := \lim_{t \rightarrow +0} [f(x_0 + tn_0) - f(x_0 - tn_0)]$$

and

$$\psi(\gamma) := \frac{2}{\pi^2} \int_0^{\min(1, \frac{1}{\gamma})} \frac{\arccos(\gamma t)}{\sqrt{1-t^2}} dt, \quad \gamma > 0; \quad \psi(-\gamma) := \psi(\gamma), \quad \gamma > 0.$$

The function $\psi(\gamma) > 0$, is monotonically decreasing on \mathbb{R}_+ , $\psi(+0) = \frac{1}{2}$, $\psi(\gamma) = \frac{2}{\pi^2 \gamma} + O\left(\frac{1}{\gamma^3}\right)$, $\gamma \rightarrow +\infty$.

If $f \in C^{k-1}(U_\rho)$, $k \geq 1$, k -th order derivatives of $f(x)$ exist in U_ρ , some of them are discontinuous across S , and S is piecewise-smooth in U_ρ , then $f_\rho^C \in C^k(U)$.

Other properties of f_ρ can be found in [1] as well as a general method for constructing a family of pseudolocal tomography functions, that is, functions which are computable from local tomographic data and having the same discontinuities and the same sizes of the jumps as $f(x)$.

References

- [1] Ramm, A.G., Katsevich, A., The Radon Transform and Local Tomography, *CRC Press*, Boca Raton 1996, pp.1-503.
- [2] Ramm, A.G., Katsevich, A., Pseudolocal tomography, *SIAM J. Appl. Math.*, 56, N1, (1996), 167-191.

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