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Obstacle scattering

Let $D \subset \mathbb{R}^3$ be a bounded domain with boundary S. A large literature, going back to mid-thirties deals with wave scattering by obstacles when S is smooth, for example, a $C^{1,\lambda}$ surface, $0 < \lambda \leq 1$. (See [3], [9], [2].)

The obstacle scattering problem consists of finding the solution to the equation:

$$(\nabla^2 + k^2)u = 0 \text{ in } D' := R^3 \backslash D, \quad k > 0, \tag{1}$$

$$\Gamma u = 0 \text{ on } S, \tag{2}$$

$$u = e^{ik\alpha\cdot x} + v, \quad \lim_{r\to\infty} \int_{|s|=r} \left| \frac{\partial v}{\partial |x|} - ikv \right|^2 ds = 0. \tag{3}$$

Here $\Gamma u = u$ (the Dirichlet condition), or $\Gamma u = u_N$ (the Neumann condition), or $\Gamma u = u_N + h(s)u$ (the Robin condition), where N is the unit normal to S pointing into D', and $h(s)$ is a continuous function. Condition (3) is the radiation condition which selects a unique solution to problem (1)-(3). In (3), $\alpha \in S^2$ is a given unit vector, the direction of the incident plane wave $e^{ik\alpha\cdot x}$, and $k > 0$ is the wave number.

The scattering problem (1)-(3) has a solution and the solution is unique.

This basic result was proved originally by the integral equations method [3].

There are many different types of integral equations which allow one to study problem (1)-(3) (see [9], where most of these equations are derived).

The scattering field v in (3) has the following asymptotics

$$v(x, \alpha, k) = \frac{e^{ikr}}{r} A(\alpha', \alpha, k) + o\left(\frac{1}{r}\right), r := |x| \to \infty, \alpha' := \frac{x}{r}. \tag{4}$$

The function $A(\alpha', \alpha, k)$ is called the scattering amplitude. This function has the following properties [9]:

i) reality: $A(\alpha', \alpha, -k) = \overline{A(\alpha', \alpha, -k)}$, $k > 0$, the bar stands for complex conjugate,

ii) reciprocity: $A(\alpha', \alpha, k) = A(-\alpha, -\alpha', k)$,

iii) unitarity: $\frac{A(\alpha',\alpha,k) - \overline{A(\alpha,\alpha',k)}}{2i} = \frac{k}{4\pi} \int_{s^2} f(\alpha', \beta, k) \overline{f(\alpha, \beta, k)} d\beta$, and its consequence, which is called the optical theorem:

$$Im A(\alpha, \alpha, k) = \frac{k}{4\pi} \int_{S^2} |f(\alpha, \beta, k)|^2 d\beta := \frac{k\sigma(\alpha)}{4\pi}$$

where $\sigma(\alpha) := \int_{S^2} |f(\alpha,\beta,k)|^2 d\beta$ is called the cross-section.

The function $A(\alpha',\alpha,k)$ is analytic with respect to k in $\mathbb{C}_+ := \{k : Imk \geq 0\}$ and meromorphic in $\mathbb{C}$, and is analytic with respect to α' and α on the variety $M := \{\theta : \theta \in \mathbb{C}^3, \theta\cdot\theta = 1\}$, where $\theta \cdot w := \sum_{j=1}^{3} \theta_j w_j$, see [9], [10].

Necessary and sufficient condition for a scatterer to be spherically summetric is: $A(\alpha',\alpha,k) = A(\alpha'\cdot\alpha,k)$, where $\alpha'\cdot\alpha$ is the dot product [10], [16].

The solution $u(x,\alpha,k)$ to (1)-(3) is called the scattering solution. Any $f(x) \in L^2(D')$ can be expanded with respect to scattering solutions:

$$\begin{cases} f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{R^3} \widehat{f}(\xi) u(x,\xi) d\xi, & \xi := k\alpha \\ \widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{D'} f(x)\overline{u(x,\xi)} dx := \mathcal{F}f \end{cases}$$

The operator $\mathcal{F} : L^2(D') \to L^2(\mathbb{R}^3)$ is unitary: $\| \mathcal{F}f \|_{L^2(\mathbb{R}^3)} = \| f \|_{L^2(D')}$, $\mathcal{F}^* = \mathcal{F}^{-1}$, see [9].

The above results hold in $\mathbb{R}^n$ with odd n. In $\mathbb{R}^n$ with even n the scattering amplitude $A(\alpha',\alpha,k)$ as a function of complex k has a logarithmic branch point at $k = 0$.

The scattering problem with minimal assumptions on the smoothness of the boundary S is studied in [22]. If $\Gamma u = u$, then existence and uniqueness of the scattering solution is proved without any assumption on the smoothness of the boundary S of a bounded domain D. In this case a weak formulation of problem (1)-(3) is considered and the limiting absorption principle is proved.

If $\Gamma u = u_N$, then again a weak formulation of (1)-(3) is considered and the only assumption on the smoothness of the boundary S is the compactness of the embedding $H^1(D'_R)$ into $L^2(D'_R)$, where $D'_R := D' \cap B_R$, and $B_R = \{x : |x| \leq R\}$ is a ball which contains D. Existence and uniqueness of the scattering solution are proved and the limiting absorption principle is established. Finally, if $\Gamma u = u_N + hu$, then the same results are obtained under the assumptions of the compactness of the embeddings $i_1 : H^1(D'_R) \to L^2(D'_R)$ and $i_2 : H^1(D'_R) \to L^2(S)$, where S is equipped with $(n-1)$-dimensional Hausdorff measure and $H^1(D'_R)$ is the Sobolev space.

For example, the embedding i_1 is compact for C-domains, that is, domains whose boundary can be covered by a finitely many open in R^3 sets and on each of these sets in a local coordinate system the equation of S can be written as $x_3 = f(x'), x' = (x_1, x_2)$, where $f(x')$ is a continuous function.

The scattering problem for one obstacle, small in comparison with the wavelength ($ka << 1$, where a is the diameter of the small obstacle), for many such bodies (the number J of bodies of order 20), and in a medium consisting of many such bodies randomly distributed in the space has been studied in [7] and later in [8]. Formulas for the scattering amplitude for acoustic and electromagnetic wave scattering by small bodies of arbitrary shapes are derived in [8].

These formulas for acoustic wave scattering on a single small body, which contains the origin, are

$$A(\alpha',\alpha,k) \equiv -\frac{C}{4\pi}, \text{ if } \Gamma u = u, \quad ka << 1,$$

where C is the electrical capacitance of the body D;

$$A(\alpha', \alpha, k) \approx -\frac{k^2 V}{4\pi}(1 + \beta_{pq}\alpha_q\alpha'_p) \text{ if } \Gamma u = u_N, \quad ka << 1,$$

where over the repeated indices one sums up from 1 to 3, v is the volume of D and $\beta_{pq} = \beta_{qp}$ is the magnetic polarizability tensor;

$$A(\alpha', \alpha, k) \approx -\frac{h|S|}{4\pi(1 + h|S|C^{-1})} \text{ if } \Gamma u = u_N + hu, \quad ka << 1, \quad h = const,$$

where $|S|$ is the area of the boundary S.

The S-matrix for the electromagnetic wave scattering by a small homogeneous body of arbitrary shape is:

$$S = \frac{k^2 V}{4\pi} \begin{pmatrix} \mu_0\beta_{11} + \alpha_{22}\cos\theta - \alpha_{32}\sin\theta & \alpha_{21}\cos\theta - \alpha_{31}\sin\theta - \mu_0\beta_{12} \\ \alpha_{12} - \mu_0\beta_{21}\cos\theta + \mu_0\beta_{31}\sin\theta & \alpha_{11} + \mu_0\beta_{22}\cos\theta - \mu_0\beta_{32}\sin\theta \end{pmatrix},$$

where β_{ij} and α_{ij} are tensors of magnetic and electic polarizability and θ is the angle between the direction e_3 of the incident field and the direction of the scattered field, μ_0 is the magnetic permeability of the medium in which the small body is emedded.

Formulas for the tensors α_{ij} and β_{ij} are derived in [7] and [8].

One can derive an equation for the average field in the medium which consists of many small obstacles (particles) randomly distributed in a region. This done in [8] and [6].

Scattering by random surfaces is studied in [1].

Inverse obstacle scattering problems are of interest:

ISP1: given $A(\alpha', \alpha_0, k)$ for all $\alpha' \in S^2$ and all $k > 0, \alpha_0 \in S^2$ is fixed, find S and the boundary condition on S.

ISP2: given $A(\alpha', \alpha, k_0)$ for all $\alpha', \alpha \in S^2, k_0 > 0$ is fixed, find S and the boundary condition on S.

ISP3: given $A(\alpha', \alpha_0, k_0)$ for all $\alpha' \in S^2, \alpha_0 \in S^2$ and $k_0 > 0$ are fixed, find S.

Uniqueness of the solution to ISP1 (for $\Gamma u = 0$) is first proved by M. Schiffer (1964) whose argument is given in [9].

Uniqueness of the solution to ISP2 is first proved by A.G. Ramm (1985) and his proof is given in [9].

Uniqueness theorem for ISP3 is not yet proved: it is an open problem to prove (or disprove) such a theorem.

One can consider inverse obstacle scattering for penetrable obstacles [23].

The Schiffer's proof of the uniqueness theorem is based on the result which says that the spectrum of the Dirichlet Laplacian in any bounded domain is discrete. This result follows from the compactness of the embedding $i : \overset{\circ}{H}{}^1(D) \to L^2(D)$ for any bounded domain (without any assumptions on the smoothness of its boundary S), $\overset{\circ}{H}{}^1(D)$ is the Sobolev space which is the closure in $H^1(D)$ of $C_0^\infty(D)$. It is known [5] that i :

$H^1(D) \to L^2(D)$ is not compact for rough domains (it is compact for Lipschitz domains, for domains satisfying cone condition, for C-domains, for E-domains, that is domains for which a bounded extension operator from $H^1(D)$ into $H^1(R^3)$ exists, see [5]).

Therefore the spectrum of a Neumann Laplacian in such a rough domain that the imbedding $i : H^1(D) \to L^2(D)$ is not compact is not discrete. One way out is given in [21] and another one in [18].

Suppose that $A_1(\alpha', \alpha, k_0)$ and $A_2(\alpha', \alpha, k_0)$ are scattering amplitudes at a fixed $k = k_0 > 0$ for two obstacles and, $sup_{\alpha', \alpha \in S^2} |A_1 - A_2| < \delta$.

Let us assume that the boundaries of the two obstacles are $C^{2,\lambda}, 0 < \lambda \leq 1$, that is, in local coordinates these boundaries $S_m, m = 1, 2$, have equations $x_3 = f_m(x_1, x_2)$, where $f \in C^{2,\lambda}$, $m = 1, 2$, $\| f_m \|_{C^{2,\lambda}} \leq c_0 = const > 0$.

Let ρ denote the Hausdorff distance between S_1 and $S_2 : \rho = \sup_{x \in S_1} \inf_{y \in S_2} |x - y|$.

The basic stability result [20] is : $\rho \leq c_1 \left(\frac{\ln |\ln \delta|}{|\ln \delta|} \right)^{c_2}$, where c_1 and c_2 are positive constants.

In [20] an open problem is formulated: derive an inversion formula for finding S given the data $A(\alpha', \alpha) := A(\alpha', \alpha, k_0) \forall \alpha'$, $\alpha \in S^2$.

Existence of such a formula is proved in [20]: if $\chi(x) := \chi_D(x)$ is the characteristic function of D and $\tilde{\chi}(\xi)$ is its Fourier transform, then there exists a function $v_\varepsilon(\alpha, \theta) \in L^2(S^2)$ such that

$$\tilde{\chi}(\xi) = \frac{8\pi}{\xi^2} \lim_{\varepsilon \downarrow 0} \int_{S^2} A(\theta', \alpha) v_\varepsilon(\alpha, \theta) d\alpha,$$

where $\theta, \theta' \epsilon M$, $\theta' - \theta = \xi$, $\xi \in \mathbb{R}^3$ is an arbitrary vector. A formula for calculation of $A(\theta', \alpha)$, $\theta' \in M$, given $A(\alpha', \alpha) \forall \alpha'$, $\alpha \in S^2$, is derived in [20]. The problem is to construct $v_\varepsilon(\alpha, \theta)$ from the data $A(\alpha', \alpha)$ algoritmically. For inverse potential scattering this is done in [17].

References

[1] Bass, F., Fuks, I., Wave scattering from statistically rough surfaces, *Pergamon Press*, New York, 1979.

[2] Colton, D., Kress, R., Integral equations methods in scattering theory, *Wiley*, New York, 1983.

[3] Kupradze, V., Randwertaufgaben der Schwingungstheorie und Integralgleichungen, *DVW*, Berlin, 1956.

[4] Leis, R., Initial boundary value problems in mathematical physics, New York, 1986.

[5] Maz'ja, V., Sobolev spaces, *Springer-Verlag*, Berlin, 1985.

[6] Marchenko, V., Khruslov, E., Boundary value problems in domains with granulated boundary, *Kiev*, Naukova Dumka, 1974.

[7] Ramm, A.G., Theory and applications of some new classes of integral equations, *Springer-Verlag*, New York, 1980.

[8] Ramm, A.G., Iterative methods for calculating the static fields and wave scattering by small bodies, *Springer Verlag*, New York, 1982.

[9] Ramm, A.G., Scattering by obstacles, *D.Reidel*, Dordrecht, 1986, pp.1-442.

[10] Ramm, A.G., Multidimensional inverse scattering problems, *Longman/Wiley*, New York, 1992, pp.1-385.

[11] Ramm, A.G., Spectral properties of the Schroedinger operator in some domains with infinite boundaries. *Soviet math. Doklady*, 152, (1963) 282-285. 27 #3930.

[12] Ramm, A.G., Reconstruction of the domain shape from the scattering amplitude. *Radiotech. i Electron.* 11, (1965), 2068-2070.

[13] Ramm, A.G., Approximate formulas for tensor polarizability and capacitance of bodies of arbitrary shape and its applications. Sov. Physics, *Doklady*, 195, 1970, 1303-1306. #1947.

[14] Ramm, A.G., Electromagnetic wave scattering by small bodies of an arbitrary shape, in the book: "Acoustic, electromagnetic and elastic scattering-Focus on T-matrix approach" Pergamon Press, N. Y. 1980. 537-546. (ed. V. Varadan).

[15] Ramm, A.G., On inverse diffraction problem. *J. Math. Anal. Appl.*, 103, (1984), 139-147.

[16] Ramm, A.G., Necessary and sufficient condition for a scattering amplitude to correspond to a spherically symmetric scatterer, *Appl. Math. Let. 2*, (1989), 263-265.

[17] Ramm, A.G., Stability estimates in inverse scattering, *Acta Appl. Math.*, 28, N1, (1992), 1-42.

[18] Ramm, A.G., New method for proving uniqueness theorems for obstacle inverse scattering problems, *Appl. Math. Lett.*, 6, N6, (1993), 19-22.

[19] Ramm, A.G., Stability estimates for obstacle scattering, *J. Math. Anal. Appl.* 188, N3, (1994), 743-751.

[20] Ramm, A.G., Stability of the solution to inverse obstacle scattering problem, *J.Inverse and Ill-Posed Problems*, 2, N3, (1994), 269-275.

[21] Ramm, A.G., Uniqueness theorems for inverse obstacle scattering problems in Lipschitz domains, *Applic. Analysis*, 59, (1995), 377-383.

[22] Ramm, A.G., Existence and uniqueness of the scattering solutions in the exterior of rough domains, (with M.Sammartino)in the book "Operator theory and applications", Amer. Math. Soc., Fields Institute Communications, Providence, 2000, pp. 457-472. (editors A.G.Ramm, P.N.Shivakumar, A.V.Strauss),

[23] Ramm, A.G., Pang, P., Yan, G., A uniqueness result for the inverse transmission problem, *Internat. Jour. of Appl. Math.*, 2, N5, (2000), 625-634.

[24] Vekua, I., Metaharmonic functions, *Trudy Tbil. Math. Inst.*, 12, (1943), 105-174.

[25] Ursell, F., On the exterior problems of acoustics, *Proc. Cambz. Phil. Soc.*, 74, (1973), 117-125; 84, (1978), 545-548.

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