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Local Tomography

Let $f(x)$ be a compactly supported piecewise-smooth function, $f(x) = 0$ if $x \notin \overline{D} \subset \mathbb{R}^2$, D is a bounded domain and $\widehat{f}(\alpha, p) = \int_{\ell_{\alpha p}} f(x)ds := Rf$ be its Radon transform, $\ell_{\alpha p} := \{x : \alpha \cdot x = p\}$ is the straight line parametrized by the unit vector α and a scalar p. The inversion formula which reconstructs $f(x)$ from the knowledge of $\widehat{f}(\alpha, p)$ for all $\alpha \in S^1$ and all $p \in \mathbb{R}$, where S^1 is the unit circle in $\mathbb{R}^2$, is known. This formula is:

$$f(x) = \frac{1}{4\pi^2} \int_{S^1} \int_{-\infty}^{\infty} \frac{\widehat{f}_p(\alpha, p)}{\alpha \cdot x - p} d\alpha dp, \quad \widehat{f}_p := \frac{\partial \widehat{f}}{\partial p} \tag{1}$$

It is nonlocal : one requires the knowlege of $\widehat{f}(\alpha, p)$ for all p in order to calculate $f(x)$.

By local tomographic data one means the values of $\widehat{f}(\alpha, p)$ for those α and p which satisfy the condition $|\alpha \cdot x_0 - p| < \delta$, where x_0 is a fixed "point of interest", and $\delta > 0$ is a small number. Geometrically local tomographic data are the valus of the integrals over the straight lines which intersect the disk centered at x_0 with radius δ. In many applications only local tomographic data are available, and in medical imaging one wants to minimize the radiation dose of a patient and to use only the local tomographic data for diagnostics.

Therefore, *the basic quesion is: what practically useful information can one get from local tomographic data?*

As mentioned above, one cannot find $f(x_0)$ from local tomographic data.

What does one mean by "practically useful information"?

By this one means the location of discontinuity curves (surfaces, if $n > 2$) of $f(x)$ and the sizes of the jumps of $f(x)$ across these surfaces.

Probably the first empirically found method for finding dicsontinuities of $f(x)$ from local tomographic data was suggested in [1], where the function

$$f_{s\ell t}(x) := -\frac{1}{4\pi} \int_{S^1} \widehat{f}_{pp}(\alpha, \alpha \cdot x) d\alpha, \tag{2}$$

which is the standard local tomography function, was proposed. To calculate $f(x)$ one needs to know only the local tomography data corresponding to the point x. It is proved that $f(x)$ and $f_{s\ell t}(x)$ have the same discontinuities (but different sizes of the jumps across the dicsontinuity curves) [11]. In [2]-[11] various aspects of local tomography are studied.

In [7]- [9] a large family of local tomography functions was proposed. The basic idea in these papers is to establish a relation between hypoelliptic pseudodifferential operators (PDO) and a class of linear operators acting on the functions $\widehat{f}(\alpha, p)$.

Let a PDO be defined by the formula $Bf = \mathcal{F}^{-1}[b(x,t,\alpha)\tilde{f}]$, where $\tilde{f} := \mathcal{F}f$ is the Fourier transform, $\tilde{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{i\xi\cdot x}dx$, and $b(x,t,\alpha)$ is a smooth function which is called the symbol of $B, \alpha := \frac{\xi}{|\xi|}, t = |\xi|$. If the symbol is hypoelliptic, that is, $c_1|\xi|^{m_1} \leq |b| \leq c_2|\xi|^{m_2}, |\xi| > R, x \in D$, c_1 and c_2 are positive constants, then $WF(Bf) = WF(f)$, where $WF(f)$ is the wave front of f. Thus the singularities of Bf and f are the same. One can prove [9] the formula $Bf = R^*(a_e \circledast \widehat{f}) := A\widehat{f}$, where $R^*g := \int_{S^{n-1}} g(\alpha, \alpha\cdot x)d\alpha, R^*$ is the adjoint to the Radon operator R, $a \circledast \widehat{f} := \int_{-\infty}^{\infty} a(x,\alpha,p-q)\widehat{f}(q)dq$ is the convolution operator, the distributional kernel $a(x,\alpha,p-q)$ is defined by the formula:

$$a(x,\alpha,p) := \frac{1}{(2\pi)^n}\int_0^\infty t^{n-1}e^{-itp}b(x,t,\alpha)dt, \tag{3}$$

and

$$a_e(x,\alpha,p) := \frac{a(x,\alpha,p) + a(x,-\alpha,-p)}{2} \tag{4}$$

is the even part of $a(x,\alpha,p)$.

An operator A is called a local tomography operator if and only if $\operatorname{supp} a_e(x,\alpha,p) \subset [-\delta,\delta]$ uniformly with respect to $x \in D$ and $\alpha \in S^{n-1}$.

A necessary and sufficient condition for A to be a local tomography operator is given in [9]: the kernel $b(x,t,\alpha)t_+^{n-1} + b(x,-t,-\alpha)t_-^{n-1}$ is an entire function of t of exponential type $\leq \delta$ uniformly with respect to $x \in D$ and $\alpha \in S^{n-1}$.

References

[1] Vainberg, E., Kazak, I., Kurczaev V., Reconstruction of the internal 3D structure of objects based on real-time integral projections, *Sov. J. Nondestr. Test.*, 17, (1981), 415-423.

[2] Faridani, A., Ritman, E., Smith, K., Local tomography, *SIAM J. Appl. Math.*, 52, (1992), 459-484.

[3] Katsevich, A., Local tomography for the generalized Radon transform. *SIAM J. Appl. Math.*, 57, No. 4, (1997), 1128-1162

[4] Katsevich, A., Local tomography for the limited-angle problem. *Journal of Mathematical Analysis and Applications*, 213, (1997), 160-182.

[5] Katsevich, A., Local tomography with nonsmooth attenuation II. In the book "Inverse Problems, Tomography, and Image Processing" (A. G. Ramm, editor), pp. 73-86, 1998, Plenum, New York.

[6] Katsevich, A., Local tomography with nonsmooth attenuation, *Transactions of the American Mathematical Society*, 351, (1999), 1947-1974.

[7] A.G Ramm, Optimal local tomography formulas, *PanAmer. Math. Journ.*, 4, N4, (1994), 125-127.

[8] A.G. Ramm, Finding discontinuities from tomographic data,*Proc. Amer. Math. Soc.*, 123, N8, (1995), 2499-2505.

[9] A.G. Ramm, Necessary and sufficient conditions for a PDO to be a local tomography operator, *Comptes Rend. Acad. Sci, Paris*, 332, N7, (1996), 613-618.

[10] A.G. Ramm New methods for finding discontinuities of functions from local tomographic data, *Jour. of Inverse and Ill-Posed Problems*, 5, N2, (1997), 165-174.

[11] A.G. Ramm, A.I. Katsevich, *The Radon Transform and Local Tomography,* CRC Press, Boca Raton 1996, pp.1-503.

A.G. Ramm
Mathematics Department
Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu