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Inverse Scattering, half-axis case

The direct scattering problem on half-axis consists of finding the solution $u(x,k)$ to the problem

$$u'' + k^2 u - q(x)u = 0, \quad x > 0, \tag{1}$$

$$u(0,k) = 0, \tag{2}$$

$$u(x,k) = e^{i\delta} \sin(kx + \delta) + o(1) \text{ as } x \to \infty. \tag{3}$$

Here $\delta = \delta(k)$ is to be determined. The function $\delta = \delta(k)$ is called the phase shift. The coefficient $q(x)$ is called the scattering potential. It is assumed to be a real-valued function in the class

$$L_{1,1} := \{q : \int_0^\infty x|q(x)dx < \infty, q = \overline{q}\},$$

the bar stands for complex conjugate. The solution $f(x,k)$ to (1) which satisfies the relation $f(x,k) = e^{ikx} + o(1)$ as $x \to +\infty$, is called the Jost solution. The function $f(0,k) := f(k)$, is called the Jost function. One has

$$f(k) = |f(k)|e^{-i\delta(k)}, \quad \delta(-k) = -\delta(k), \quad k \in \mathbb{R}, \quad \delta(\infty) = 0,$$

if $q \in L_{1,1}$ then $f(x,k)$ exists and is unique, $f(k)$ is analytic in $\mathbb{C}_+ := \{k : Imk > 0\}$ and has at most finitely many zeros in $\mathbb{C}_+$, all of which are simple and of the form ik_j, $k_j > 0$, $1 \leq j \leq J$. The numbers $-k_j^2$ are the eigenvalues of the selfadjoint operator $l := -\frac{d^2}{dx^2} + q(x)$ which is determined by the Dirichlet boundary condition at $x = 0$ in the Hilbert space $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$. In physics $-k_j^2$ are called the bound states. The positive numbers $s_j :=\| f(x, ik_j) \|^{-2}_{L^2(\mathbb{R}_+)}$ are called the norming constants. The function $S(k) := \frac{f(-k)}{f(k)} = e^{2i\delta(k)}$ is called the S-matrix. The triple $\mathcal{S} := \{S(k), ik_j, s_j, 1 \leq j \leq J\}$ is called the scattering data.

The inverse scattering problem (ISP) consists of finding $q(x)$ given $\mathcal{S}$.

The point $k = 0$ can also be a zero of $f(k)$. It is called a resonance at $k = 0$. If $f(0) = 0$ then $f'(0) \neq 0$. The basic results of the inverse scattering theory are (see [5], [6]):

1) The uniqueness theorem: $\mathcal{S} \Rightarrow q$, *that is the scattering data determined* $q \in L_{1,1}$ *uniquely.*

2) The reconstruction theorem: *If $\mathcal{S}$, corresponding to a $q \in L_{1,1}$, is given, then $q(x)$ can be reconstructed as follows:*

Step 1. Calculate $F(x) := \sum_{j=1}^{J} s_j e^{-k_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}[1 - S(k)]e^{ikx}dk.$

Step 2. Solve the equation for $A(x, y)$:

$$A(x,y) + F(x,y) + \int_x^\infty A(x,s)F(s+y)ds = 0, \quad y \geq x \geq 0, \tag{4}$$

which is uniquely solvable. It is called the Marchenko equation.

Step 3. Calculate $q(x) = -2\frac{dA(x,x)}{dx}$.

3) The characterization theorem: *For $\mathcal{S}$ to be the scattering data corresponding to a $q \in L_{1,1}$ it is necessary and sufficient that the following conditions hold:*

i) $\overline{S(k)} = S(-k) = S^{-1}(k)$, $k \in \mathbb{R}_+$; $S(\infty) = 1$, $k_j > 0$, $s_j > 0$, $1 \leq j \leq J$;

ii) $ind\, S(k) = -\kappa, \kappa = 2J$ or $\kappa = 2J + 1$;

iii) $\| F(x) \|_{L^\infty(\mathbb{R}_+)} + \| F(x) \|_{L^1(\mathbb{R}_+)} + \| xF'(x) \|_{L^1(\mathbb{R}_+)} < \infty$.

Here $ind\, S(k) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\ln S(k)$.

Note that $\kappa = 2J$ if $f(0) \neq 0, \kappa = 2J + 1$ if $f(0) = 0$. The map $T : q \to \mathcal{S}$ is a homeomorphism between $L_{1,1}$ and the space of the scattering data equipped with the norm $\| \mathcal{S} \| := \int_0^\infty (1 + x)|F'(x)|dx$ (see [4] - [6]).

One can prove (see [6], [13]) the diagram

$$\delta \Leftrightarrow F \Leftrightarrow A \Leftrightarrow q,$$

each step of which is invertible. Here $F = F(x)$ and $A = A(x, y)$ are defined above. This result guarantees in particular that the potential recovered by the Marchenko method generates the original scattering data (provided that $q \in L_{1,1}$ or $\mathcal{S}$ satisfies the characterization conditions).

Other methods for solving the ISP on the half-axis are based on the solution of inverse problem of recovery of $q(x)$ from the spectral function $\rho := \rho(\lambda)(\mathcal{S} \Rightarrow \rho \Rightarrow q)$ and the Krein method ([1], [3], [5] [6], [15]).

The scattering data are in one-to-one correspondence with the spectral function [6] [7], [13]. Recovery of $q(x)$ given the spectral function is discussed in [1], [3], [5], [6].

The original work of Krein [2] and its review in [1] do not contain proofs. A detailed presentation of Krein's theory with complete proofs is given in [15] for the first time. Also a proof of consistency of Krein's method is given in [15]. In [2] (and in [1]) there is no discussion of the consistency of Krein's method. By the consistency of an inversion method one means a proof of the implication $q \Rightarrow \mathcal{S}$ (the reconstructed potential generates the data from which is was reconstructed).

Let us describe Krein's method under the simplifying assumption $\kappa = 0$ (no bound states and no resonance at $k = 0$). The general case is treated in [15].

Step 1. Given $S(k) = e^{2i\delta(k)}$, $ind\, S(k) = 0$, $\delta(\infty) = 0$, one finds $\delta(k)$, then calculates

$$g(t) := -\frac{2}{\pi}\int_0^\infty \delta(k) sin(kt) dk, \quad f(k) = \exp(\int_0^\infty g(t)e^{ikt}dt),$$

and

$$H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (|f(k)|^{-2} - 1)e^{-ikt}dk.$$

Step 2. Given $H(t)$ one solves the equation

$$(I + H_x)\Gamma_x := \Gamma_x(t,s) + \int_0^x H(t-u)\Gamma_x(u,s)du = H(t-s), \quad 0 \le t, s \le x,$$

for $\Gamma_x(t,s)$ and finds $\Gamma_{2x}(2x,0)$, $0 \le x < \infty$.

Step 3. One defines $A(x) = 2\Gamma_{2x}(2x,0)$, and calculates $q(x) = A^2(x) + A'(x)$. Alternatively, $q(x) = 2\frac{d}{dx}[\Gamma_{2x}(2x,0) - \Gamma_{2x}(0,0)]$.

In Step 1, one can find $f(k)$ by a different method: solve the Riemann problem

$$\varphi_+(k) = S(-k)\varphi_-(k), k \in \mathbb{R}, \quad \varphi_\pm(\infty) = 1. \tag{5}$$

If $ind\, S(k) = 0$, this problem has the unique solution $\{\varphi_+(k), \varphi_-(k)\}$. One has $\varphi_+(k) = f(k), \varphi_-(k) = f(-k)$.

Note that the data $\mathcal{S}$ allow one to find a unique $f(k)$ by solving the Riemann problem (5) with the additional conditions: $\varphi_+(k)$ has J simple zeros at the points ik_j if $\kappa = -2J$ and, if $\kappa = -2J - 1, \varphi_+(k)$ has additionally a simple zero at $k = 0$. Thus, the data $\mathcal{S}$ is equivalent to the data $\{f(k), s_j, 1 \le j \le J\}$.

An inverse problem of recovery of $q(x)$ from incomplete scattering data but with an a priori assumption that $q(x)$ has compact support is investigated in [8] [9]. It is proved that if $q \in L_{1,1}$ is compactly supported and if $\delta(k)$ is known for a sequence $k = k_n > 0$ which has a finite limit point inside $(0, \infty)$, then $q(x)$ is determined uniquely. An algorithm for finding a compactly supported $q(x)$ from $\delta(k)$ (that is, from $\mathcal{S}(k)$), known for all $k > 0$ is given in [8]. Uniqueness theorem for the problem of finding a compactly supported $q(x)$ from the knowledge of $f'(0,k)$ $\forall k > 0$ is proved in [13].

In [7], [12] and algorithm for recovery of $q(x)$ from the I-function is given, where the I-function is identical with the Weyl function.

For $q \in L_{1,1}$ to belong to $L^2(\mathbb{R}_+)$ it is necessary and sufficient that ([7])

$$k\left[1 - S(k) + \frac{Q}{ik}\right] \in L^2(\mathbb{R}),$$

where

$$Q := \int_0^\infty q(t)dt = -2i \lim_{k\to\infty} \{k[f(k) - 1]\}.$$

If $q(x) \in L_{1,1} \cap L^2(\mathbb{R}_+)$, $q = 0$ for $x \geq a$, is compactly supported then $f(k)$ is an entire function of exponential type $\leq 2a$. Its zeros in $\mathbb{C}_- := \{k : Imk < 0\}$ are called resonances.

If $q(x) \not\equiv 0$, $\int_0^\infty x^n |q(x)| dx = o(n^{bn})$, $0 \leq b < 1$, then there are infinitely many resonances [7].

There exists a $q(x) \in C_0^\infty(\mathbb{R}_+)$, $q(x) = 0$ for $x \geq \epsilon$, where $\epsilon > 0$ is arbitrary small, which generates infinitely many purely imaginary resonances [6].

If $q(x) \in L_{1,1}$, $q(x) = 0$ for $x \geq a$ and $q(x)$ does not change sign in an interval $(a-\delta, a)$, where $\delta > 0$ is arbitrarily small, then $q(x)$ generates only finitely many purely imaginary resonances [6].

If $q \in L_{1,1}$, then the following estimate (see [5]) is useful:

$$\left| F'(2x) - \frac{q(x)}{4} + \frac{1}{4}\left(\int_x^\infty q(t)dt\right)^2 \right| \leq c\sigma^2(x), \quad \sigma(x) := \int_x^\infty |q(t)|dt.$$

The Jost solution $f(x,k)$ can be written as $f(x,k) = e^{ikx} + \int_x^\infty A(x,y)e^{iky}dy$, where $A(x,y)$ is the kernel of the transformation operator. If $q \in L_{1,1}$, then

$$|F(2x) + A(x,x)| \leq c\sigma(x), |F(2x) \leq c\sigma(x), \quad |A(x,y)| \leq c\sigma\left(\frac{x+y}{2}\right),$$

$$\left|\frac{\partial A(x,y)}{\partial x} + \frac{1}{4}q\left(\frac{x+y}{2}\right)\right| \leq c\sigma(x)\sigma\left(\frac{x+y}{2}\right),$$

$$\left|\frac{\partial A(x,y)}{\partial y} + \frac{1}{4}q\left(\frac{x+y}{2}\right)\right| \leq c\sigma(x)\sigma(\frac{x+y}{2}),$$

where $c > 0$ is a constant. The function $A(x,y)$ solves the following Volterra-type equation:

$$A(x,y) = \frac{1}{2}\int_{\frac{x+y}{2}}^\infty q(t)dt + \int_{\frac{x+y}{2}}^\infty ds \int_0^{\frac{y-x}{2}} q(s-t)A(s-t,s+t)dt.$$

If $q \in L_{1,1}$ and $q(x) = 0$ for $x > a$, then $A(x,y) = 0$ for $y \geq x \geq a, F(x) = 0$ for $x = 2a$, and $A(y) := A(0,y) = 0$ for $y \geq 2a$. Since $f(k) = 1 + \int_0^\infty A(y)e^{iky}dy$, it follows that $f(k)$ is an entire function of order 1 and type $\leq 2a$, and $S(k) = \frac{f(-k)}{f(k)}$ is meromorphic on the whole complex k-plane.

Conversely, if the scattering data $\mathcal{S}$ correspond to a $q \in L_{1,1}$ (necessary and sufficient conditions for this were given above) and generate (by solving the Riemann problem mentioned above) the function $f(k)$ which is an entire function of exponential type $\leq 2a$, then $q(x) = 0$ for $x > a$, (see [6]).

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