Encyclopedia of Mathematics, Supplemental Vol. 3, Kluwer Acad. Publishers, Dordrecht, 2001, 207-208

Inverse scattering, full line case

Let $q(x) \in L_{1,1} := \{q : \int_{-\infty}^{\infty}(1+|x|)|q(x)|dx < \infty, q = \overline{q}\}$, the bar stands for complex conjugate. Consider the (direct) scattering problem:

$$\ell u_- - k^2 u_- := (-u'' + q(x) - k^2)u_- = 0, \quad x \in \mathbb{R} := (-\infty, \infty), \tag{1}$$

$$u_- = \begin{cases} e^{-ikx} + r_-(k)e^{-ikx}, & x \to -\infty, \\ t_-(k)e^{ikx}, & x \to +\infty. \end{cases} \tag{2}$$

The coefficients $r(k)$ and $t(k)$ are called the reflection and transmission coefficients. One can prove that $t_-(k)$ is analytic in $\mathbb{C}_+ := \{k : Imk > 0\}$ except at a finite number of points ik_j, $1 \leq j \leq J$, $k_j > 0$, which are simple poles of $t(k)$.

Problem $(1) - (2)$ describes scattering by a plane wave e^{ikx} falling from $-\infty$ and scattered by the potential $q(x)$.

One can also consider the scattering of the plane wave falling from $+\infty$:

$$\ell u_+ - k^2 u_+ = 0, \quad x \in \mathbb{R}, \tag{3}$$

$$u_+ = \begin{cases} t_+(k)e^{-ikx}, & x \to +\infty \\ e^{-ikx} + r_+ e^{ikx}, & x \to -\infty. \end{cases} \tag{4}$$

One proves that $t_-(k) = t_+(k) := t(k)$, $t(-k) = \overline{t(k)}$, $k \in \mathbb{R}$, the bar stands for complex conjugate, $r_\pm(-k) = \overline{r_\pm(k)}$. The matrix $\begin{pmatrix} t(k) & r_-(k) \\ r_+(k) & t(k) \end{pmatrix} = S(k)$ is called the S-matrix. The energy conservation implies $|t(k)|^2 + |r(k)|^2 = 1$.

Let $f(x,k)$ and $g(x,k)$ be the solutions to (1) satisfying the conditions:

$$f(x,k) = e^{ikx} + o(1), \quad x \to \infty, \quad g(x,k) = e^{-ikx} + o(1), \quad x \to - - \infty.$$

Then

$$f(x,k) = e^{ikx} + \int_x^\infty A_+(x,y)e^{iky}dy,$$

$$g(x,k) = e^{-ikx} + \int_{-\infty}^x A_-(x,y)e^{-iky}dy,$$

where $A_\pm(x,y)$ are the kernels which define the transformation operators. One has

$$f(x,k) = b(k)g(x,k) + a(k)g(x,-k),$$

$$g(x,k) = -b(-k)f(x,k) + a(k)f(x,-k),$$

where

$$a(-k) = \overline{a(k)}, \quad b(-k) = \overline{b(k)}, \quad |a(k)|^2 = 1 + |b(k)|^2,$$
$$r_-(k) = \frac{b(k)}{a(k)}, \quad r_+(k) = -\frac{b(-k)}{a(k)}, \quad t(k) = \frac{1}{a(k)}.$$

The function $a(k)$ is analytic in $\mathbb{C}_+ := \{k : Imk > 0\}$ and has finitely many simple zeros all of which are at the points ik_j, $1 \le j \le J$, $a(ik_j) = 0$, $\dot{a}(ik_j) \ne 0$, $\dot{a} := \frac{da}{dk}$. If $k = ik_j$, then $f(x, ik_j) \in L^2(\mathbb{R})$,

$$-f''(x,ik_j) + q(x)f(x,ik_j) + k_j^2 f(x,ik_j) = 0,$$
$$\int_{-\infty}^{\infty} |f(x,ik_j)|^2 dx = (m_j^+)^{-2}_,$$
$$\int_{-\infty}^{\infty} |g(x,ik_j)|^2 dx = (m_j^-)^{-2}_.$$

The numbers $-k_j^2$ are the eigenvalues of the operator $-\frac{d^2}{dx^2} + q(x)$ in $L^2(\mathbb{R})$. They are called the bound states.

The scattering data are the values

$$\mathcal{S} := \{r_+(k), ik_j, (m_j^+)^2, \quad \forall k > 0, \quad 1 \le j \le J\}.$$

The inverse scattering problem (ISP) consists of finding $q(x) \in L_{1,1}$ from $\mathcal{S}$.

The ISP has at most one solution in the class $L_{1,1}$. This solution can be calculated by the following Marchenko method:

Step 1. Define

$$F_+(x) = \sum_{j=1}^{J} (m_j^+)^2 e^{-k_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r_+(k) e^{ikx} dk \tag{5}$$

and solve the Marchenko equation for $A_+(x,y)$:

$$A_+(x,y) + F_+(x+y) + \int_x^{\infty} A(x,t)F_+(t,y)dt = 0, \quad y \ge x$$

If the data $\{r_+(k),\ ik_j,\ (m_j^+)^2,\ 1 \le j \le J\}$ correspond to a $q \in L_{1,1}$, then equation (5) is uniquely solvable in $L^1(x,\infty)$ for every $x > -\infty$.

Step 2. If $A_+(x,y)$ is found, then $q(x) = -2\frac{dA_+(x,x)}{dx}$.

The main result [7] is the characterization property for the scattering data:

In order that $\mathcal{S} := \{r_+(k),\ ik_j,\ (m_j^+)^2,\ 1 \le j \le J,\ k_j > 0,\ m_j^+ > 0,\ k > 0\}$ be the scattering data corresponding to a $q(x) \in L_{1,1}(\mathbb{R})$, it is necessary and sufficient that the following conditions hold:

i) $r(-k)=\overline{r(k)}$ for $k>0$; the function $r(k)$ for $k\neq 0$ is continuous, $|r_+(k)|\leq 1-ck^2(1+k^2)^{-1}$, $c=const>0$, $r_+(k)=O(\frac{1}{k})$ as $k\to\pm\infty$;

ii) The function $R_+(x):=\frac{1}{2\pi}\int_{-\infty}^{\infty}r_+(k)e^{ikx}dk$ is absolutely continuous and

$$\int_s^{\infty}|R'_+(x)|(1+|x|)dx<\infty \quad \text{for every} \quad s>-\infty.$$

iii) Denote

$$a(z):=\prod_{j=1}^{J}\frac{z-ik_j}{z+ik_j}exp\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ln(1-|r_+(k)|^2)}{k-z}dk\}.$$

The function $za(z)$ is continuous in $\overline{\mathbb{C}_+}$ and

$$lim_{k\to 0}ka(k)[r_+(k)+1]=0$$

iv) The function

$$R_-(x):=-\frac{1}{2\pi}\int_{-\infty}^{\infty}r_+(-k)\frac{a(-k)}{a(k)}e^{-ikx}dk$$

is absolutely continuous and

$$\int_s^{\infty}(1+|x|)|R'_-(x)|dx<\infty \quad \text{for every} \quad s>-\infty.$$

A similar result holds for the data $\{r_-(k), ik_j, (m_j^-)^2, 1\leq j\leq J, \ \forall k>0\}$ and the potential $q(x)$, can obtained by the Marchenko method, $q(x)=-2\frac{dA_-(x,x)}{dx}$.

In [4] the above theory is generalized to the case when $q(x)$ tends to a different constants as $x\to+\infty$ and $x\to-\infty$.

In [5] a different approach to solving ISP is described for $q\in L_{1,2}:=\{q: q=\overline{q}, \int_{-\infty}^{\infty}(1+x^2)|q(x)|dx<\infty\}$.

The approach in [5] is based on a trace formula.

If $q(x)=0$ for $x<x_0<\infty$, then the reflection coefficient $\{r_+(k), \ \forall k>0\}$ alone, without the knowledge of ik_j and $(m_j^+)^2$, determines $q(x)$ uniquely. A simple proof of this and similar statements, based on property C for ODE, is given in [10].

An inverse scattering problem for an inhomogeneous Schrödinger equation is studied in [5].

Inverse scattering method is a tool for solving many nonlinear evolution equations and soliton theory [7], [1], [2], [6].

Methods for adding and removing bound states are described in [5]. They are based on the Darboux-Crum transformations and commutation formulas.

A large bibliography is in [3].

References

[1] Ablowitz, M., Segur, H., Solutions and inverse scattering transform, *SIAM*, Philadelphia, 1981.

[2] Calogero, F., Degasperis, A., Solutions and the spectral transform, North Holland, Amsterdam, 1982.

[3] Chadan, K., Sabatier, P., Inverse problems is quantum scattering, Springer Verlag, Berlin, 1989

[4] Cohen, A., Kappeler, T., Scattering and inverse scattering for step-like potentials in the Schrödinger equation, *Indiana Math. J.* 34,(1985), 127-180.

[5] Deift, P., Trubowitz, E., Inverse scattering on the line, *Comm. Pure Appl. Math.*, 32, (1979), 121-251.

[6] Faddeev, L., Takhtadjian, L., *Hamiltonian methods in the theory of solutions*, Springer-Verlag, Berlin, 1986.

[7] Marchenko, V., *Steurm-Liouville Operators and applications*, Birkhäuser, Basel, 1986.

[8] A.G. Ramm, Multidimensional inverse scattering problems, *Longman/Wiley*, New York, 1992, pp.1-385.

[9] A.G. Ramm, Inverse problem for an inhomogeneous Schrödinger equation, *Jour. Math. Phys*, 40, N8, (1999),3876-3880.

[10] A.G. Ramm, Property C for ODE and applications to inverse problems, in the book "Operator theory and applications", *Amer. Math. Soc.*, Fields Institute Communications, Providence, 2000, pp. 15-75.

A.G. Ramm
Mathematics Department
Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu