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Inverse scattering: multidimensional case.

There are many miltidimensional inverse scattering problems. Quite schematically, let us discuss inverse potential scattering (IPS), inverse geophysical scattering (IGS) and inverse obstacle scattering (IOS) problems.

To formulate IPS, consider first the direct scattering problem (see [1], [2], [4], [5], [7], Appendix):

$$[-\nabla^2 + q(x) - k^2]u = 0 \text{ in } R^3, \quad k = const > 0, \tag{1}$$

$$u = e^{ik\alpha\cdot x} + v, \quad \alpha \in S^2, \tag{2}$$

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial v}{\partial r} - ikv \right|^2 ds = 0, \tag{3}$$

where α is given, S^2 is the unit sphere, v is the scattered field, u is the scattering solution, condition (3) is called the (outgoing) radiation condition, $e^{ik\alpha\cdot x}$ is the incident plane wave, $q(x)$ is a real-valued function, which is called a potential,

$$q(x) \in L^2_{loc}(\mathbb{R}^3), \quad |q(x)| \le c(1+|x|)^{-b}, \quad b > 2, \text{ for large } |x|.$$

The existence and uniqueness of the solution to (1)-(3) was proved under less restrictive assumptions on $q(x)$ [2]. The function v has the form

$$v(x,\alpha,k) = A(\alpha',\alpha,k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r\to\infty, \quad \frac{x}{r} = \alpha',$$

where the coefficient $A(\alpha',\alpha,k)$ is called the scattering amplitude.

The IPS problem consists of finding $q(x)$ given $A(\alpha',\alpha,k)$ on some subsets of $S^2 \times S^2 \times \mathbb{R}_+$.

The first result is simple: *if $A(\alpha',\alpha,k)$ is known for all $\alpha',\alpha \in S^2$ and all $k > 0$, then $q(x)$ is uniquely determined.*

If

$$q \in Q_m := \{q : q = \overline{q}, |q(x)| + |\nabla^m q| \le c(1+|x|)^{-b}, \quad b > 3\}$$

then it is known (e.g. [7], p. 233, see also [4]) that

$$A(\alpha',\alpha,k) = -\frac{1}{4\pi}\int_{R^3} e^{ik(\alpha-\alpha')\cdot x} q(x)dx + O\left(\frac{1}{k}\right), \quad k\to\infty,$$

so that $\widetilde{q}(\xi) := \int_{R^3} e^{-i\xi\cdot x} q(x) dx$ can be found:

$$\widetilde{q}(\xi) = -4\pi \lim_{\substack{k\to\infty \\ k(\alpha-\alpha')=\xi}} A(\alpha', \alpha, k).$$

The second result is much more difficult.

For decades it was not known if the data $A(\alpha', \alpha) := A(\alpha', k_0)\ \forall\alpha', \alpha \in S^2$ and $k_0 > 0$ fixed determine $q(x)$ uniquely. In 1987 the uniqueness result has been established by Ramm (see [8], [6]) under the assumptions $q(x) \in L^2(\mathbb{R}^3)$, $q(x) = 0$ for $|x| > a$, where $a > 0$ is an arbitrary large fixed number, and in 1988 inversion procedures were published. They are described in [8]. One of them, proposed by Ramm, is based on the formula

$$\widetilde{q}(\xi) = -4\pi \lim_{\substack{|\theta|\to\infty \\ \theta,\theta'\in M \\ \theta-\theta'=\xi}} \int_{S^2} A(\theta', \alpha) v(\alpha, \theta) d\alpha,$$

where $M := \{\theta : \theta \in \mathbb{C}^3, \theta\cdot\theta = k_0^2\}$, $\theta\cdot w := \sum_{j=1}^3 \theta_j \cdot w_j$, $v(\alpha, \theta) \in L^2(S^2)$, and $\xi \in R^3$ is an arbitrary point.

Another inversion procedure ([3], [8]) is based on the reconstruction of the Dirichlet-to-Neumann map and then finding $q(x)$.

Error estimates for the Ramm's inversion procedure in the case of noisy data are obtained in [9].

The uniqueness problem for IPS with the data $A(\alpha', \alpha_0, k)\ \forall\alpha' \in S^2, \forall k > 0, \alpha_0 \in S^2$, fixed, is still open.

The same is true for the uniqueness problem for IPS with the data $A(-\alpha, \alpha, k)\ \forall\alpha \in S^2, \forall k > 0$, (backscattering data) although for this problem uniqueness theorem for small $q(x)$ holds.

The IGS problem consists of finding the unknown coefficient $v(x)$ in the equation

$$(\nabla^2 + k_0^2 + k_0^2 v(x)) u(x, y, k_0) = -\delta(x - y) \text{ in } \mathbb{R}^3, \tag{4}$$

$u := u(x, y) := u(x, y, k_0)$ satisifies the outgoing radiation condition (3), $k_0 = const > 0$ is fixed, $v(x)$ is a real-valued L^2_{loc} function with compact support in $\mathbb{R}^3_- := \{x : x_3 < 0\}$.

The scattering data are the values $u(x, y) \forall x, y \in P := \{x : x_3 = 0\}$, that is, the values of u on the surface of the Earth. The function $v(x)$ describes an inhomegeneity in the velocity profile (in the refraction coefficient), u can be an acoustic pressure. Uniqueness of the solution to IGS problem was proved by Ramm (1987) (see [8]).

The uniqueness problem for IGS with data $u(x, y_0, k)\ \forall x \in P, \forall k > 0$, and $y_0 \in P$ fixed, is open.

A reduction of the IGS problem with the data $u(x, y, k_0)\ \forall x, y \in P$. To the IPS problem with the data $A(\alpha', \alpha, k_0)\ \forall\alpha, \alpha' \in S^2_+$, $k_0 > 0$ fixed, $S^2_+ := \{\alpha : \alpha \in S^2, \alpha\cdot e_3 > 0\}$, e_3 is the unit vector along x_3-axis, is done in [8].

For the IOS problem see the entry obstacle scattering.

An interesting open problem in IPS is the problem of finding discontinuities of $q(x)$ and the number of bound states of the Schrödinger operator, generated by the experssion $-\nabla^2 + q(x)$ in $L^2(\mathbb{R}_3)$, from the knowledge of fixed energy scattering data $A(\alpha', \alpha, k_0)$, $\forall \alpha', \alpha \in S^2$.

If $q \in L_0^2(\mathbb{R}^3)$ then $A(\alpha', \alpha)$ is an analytic function of α', $\alpha \in M$. Therefore, the knowledge of $A(\alpha', \alpha)$ on an open set in $S^2 \times S^2$, however small, allows one to recover $A(\alpha', \alpha)$ on $M \times M$.

The assumption concerning compactness of the support of $q(x)$ is natural in IPS because the scattering data are always noisy and it is not possible in principle to recover the tale of a $q(x) \in Q$ (that is $q(x)$ for $|x| > R$, where $R > 0$ is sufficiently large), from the knowledge of noisy data $A_\delta(\alpha', \alpha)$, $\sup_{\alpha, \alpha' \in S^2} |A_\delta(\alpha', \alpha) - A(\alpha', \alpha)| < \delta$ (see [8] for a proof).

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