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Partial Differential Equations, Property C for

Property C stands for completeness of the set of products of solutions to homogeneous linear partial differential equations. It was introduced in [1] and used in [2]-[12] as a powerful tool for proving uniqueness results for many multidimensional inverse problems, in particular, inverse scattering problems.

To define property C let us introduce some notations. Let D be a bounded domain in $\mathbb{R}^n, n \geq 2, L_m u(x) := \sum_{|j|=0}^{J} a_{jm}(x) D^j u(x)$, where j is a multiindex, $D^j u = \frac{\partial^{|j|}u}{\partial x_1^{j_1} \ldots \partial x_n^{jn}}$, derivative are understood in the distributional sense, $a_{jm}(x)$, $m = 1, 2$, are some $L^\infty(D)$ functions, $N_m := \{w : L_m w = 0 \text{ in } D\}$ is the null-space of the formal differential operator L_m, the equation $L_m w = 0$ is understood in the distributional sense.

Consider the subsets $\tilde{N}_1 \in N_2$ and $\tilde{N}_2 \in N_n$ for which the products $w_1 w_2$ are defined, $w_1 \in \tilde{N}_1, w_2 \in \tilde{N}_2$.

Definition 1. *The pair $\{L_1, L_2\}$ has property C_p iff the set $\{w_1 w_2\}_{\forall w_m \in \tilde{N}_m}$ is total (complete) in $L^p(D)$, ($p \geq 1$ is fixed), that is, if $f(x) \in L^p(D)$ and*

$$\int_D f(x) w_1(x) w_2(x) dx = 0 \quad \forall w_1 \in \tilde{N}_1, \quad \forall w_2 \in \tilde{N}_2, \tag{1}$$

then $f(x) \equiv 0$.

By property C we mean often property C_2 or C_p with any fixed $p \geq 1$. Is property C generic for a pair of formal partial differential operators L_1 and L_2?

For the operators with constant coefficients necessary and sufficient condition is given in [10] for a pair $\{L_1, L_2\}$ to have property C. For such operators it turns out that property C is generic and holds or fails to hold simultaneously for all $p \in [1, \infty)$.

Let us formulate the result from [10]. Assume $a_{jm}(x) = a_{jm} = const$. Denote $L_m(z) := \sum_{|j|=0}^{J} a_{jm} z^j, z \in \mathbb{C}^n$. Note that $L_m(e^{z\cdot x}) = e^{z \cdot x} L_m(z)$, $z \cdot x := \sum_{j=1}^{n} z_j x_j$.

Therefore $e^{z\cdot x} \in \tilde{N}_m$ if and only if $L_m(z) = 0$.

Define algebraic varieties

$$\mathcal{L}_m := \{z : z \in \mathbb{C}^n, L_m(z) = 0\}. \tag{2}$$

We say that $\mathcal{L}_1$ is transversal to $\mathcal{L}_2$ and write $\mathcal{L}_1 \not\parallel \mathcal{L}_2$ if and only if there exist a point $\zeta \in \mathcal{L}_1$ and a point $\xi \in \mathcal{L}_2$ such that the tangent space T_1 to $\mathcal{L}_1$ (in $\mathbb{C}^n$) at the point ζ and the tangent space T_2 to $\mathcal{L}_2$ at the point ξ are transversal.

The following result is proved in [1]: the pair $\{L_1, L_2\}$ of the formal partial differential operators with constant coefficients has property C if and only if $\mathcal{L}_1 \nparallel \mathcal{L}_2$.

Thus, property C fails to hold for a pair $\{L_1, L_2\}$ of formal differential operators with constant coefficients if and only if the variety $\mathcal{L}_1 \cup \mathcal{L}_2$ is a union of parallel hyperplanes in $\mathbb{C}^n$.

Therefore, property C for partial differential operators with constant coefficients is generic.

Definition 2. *If $L_1 = L_2 = L$ and the pair $\{L, L\}$ has property C, then we say that L has property C.*

Examples.

Let $n \geq 2, L = \nabla^2 := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. Then $L = \{z : z \in \mathbb{C}^n, z_1^2 + \cdots + z_n^2 = 0\}$. It is easy to check that there are points $\zeta \in \mathcal{L}$ and $\xi \in \mathcal{L}$ at which the tangent hyperplanes to $\mathcal{L}$ are not parallel. Thus $L = \nabla^2$ has property C. This means that the set of products of harmonic functions in a bounded domain $D \subset \mathbb{R}^n$ is complete in $L^p(D), p \geq 1$. Similarly one checks that the operators

$$L = \frac{\partial}{\partial t} - \nabla^2, \quad L = \frac{\partial^2}{\partial t^2} - \nabla^2, \quad L = i\frac{\partial}{\partial t} - \nabla^2 \tag{3}$$

have property C.

Numerous applications of property C to inverse problems can be found in [1].

Property $C = C_2$ holds for a pair of Schrödinger operators with potentials $q_m(x) \in L_0^2(R^n), n \geq 3$, where $L_0^2(R^n)$ is the set of $L^2(R^n)$ functions with compact support.

If $u_m(x, \alpha, k), m = 1, 2, \alpha \in S^{n-1}, k = \mathit{const} > 0$, S^{n-1} is the unit sphere in $\mathbb{R}^n$, are the **scattering solutions** corresponding to the Schrödinger operators $l_m = -\nabla^2 + q_m(x) - k^2, q_m(x) \in L_0^2(R^n), n \geq 3$, then the set of products $\{u_1(x, \alpha, k)u_2(x, \beta, k)\}_{\forall \alpha, \beta \in S^{n-1}}$, $k = \mathit{const} > 0$ is fixed, is complete in $L^2(D)$, where $D \subset \mathbb{R}^n$ is an arbitrary fixed bounded domain [1]. The set $\{u_m(x, \alpha, k)\}_{\forall \alpha \in S^{n-1}}$, where $k > 0$ is fixed, is total in the set $N_m := \{w : l_m w = 0 \text{ in } D, w \in H^2(D)\}$, where $H^2(D)$ is the **Sobolev space** [1].

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