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Ordinary differential equations, property C for

Let

$$\ell_m u = \left(-\frac{d^2}{dx^2} + q_m(x) \right) u, \quad m = 1, 2, \quad x \in \mathbb{R}_+ := [0, \infty),$$

$q_m(x)$ is a real-valued function,

$$q_m(x) \in L_{1,1}(\mathbb{R}_+) := \left\{ q : \int_0^\infty x|q(x)|dx < \infty \right\}.$$

Consider the problem

$$(\ell_m - k^2) f_m = 0, \quad x \in \mathbb{R}_+, \quad f_m(x, k) = e^{ikx} + o(1) \text{ as } x \rightarrow +\infty.$$

This problem has a unique solution which is called the **Jost function**. Define also the solution to the problem:

$$(\ell_m - k^2) \varphi_m(x, k) = 0 \quad x \in \mathbb{R}_+, \quad \varphi_m(0, k) = 0, \quad \varphi'_m(0, k) = 1,$$

and to the problem

$$(\ell_m - k^2) \psi_m(x, k) = 0 \quad x \in \mathbb{R}_+, \quad \psi_m(0, k) = 1, \quad \psi'_m(0, k) = 0.$$

Assume $h(x) \in L^2(\mathbb{R}_+)$ and

$$\int_0^\infty h(x) f_1(x, k) f_2(x, k) dx = 0 \quad \forall k > 0. \quad (*)$$

Definition 1. If $(*)$ implies $f(x) \equiv 0$, then we say that the pair $\{\ell_1, \ell_2\}$ has property C_+ .

Let $b > 0$ be an arbitrary fixed number, $h(x) \in L^1(\mathbb{R}_+)$ and assume

$$\int_0^b h(x) \varphi_1(x, k) \varphi_2(x, k) dx = 0 \quad \forall k > 0. \quad (**)$$

Definition 2. If $(**)$ implies $h(x) \equiv 0$, then we say that the pair $\{\ell_1, \ell_2\}$ has property C_φ .

Similarly one defines property C_ψ .

It is proved in [1] that the pair $\{l_1, l_2\}$ has property C_+ if $q_m \in L_{1,1}, m = 1, 2$.

It is proved in [2] that the pair $\{l_1, l_2\}$ has properties C_φ and C_ψ .

However, if $b = \infty$, then, in general, property C_φ fails to hold for a pair $\{l_1, l_2\}$. This means that there exist $h(x) \not\equiv 0, h \in L^1(\mathbb{R}_+)$, and two potentials $q_1, q_2 \in L_{1,1}$, such that (*) holds for all $k > 0$.

In [2] many applications of properties C_+, C_φ and C_ψ to inverse problems are presented.

For instance, suppose that the I -function, defined as $I(k) := \frac{f'(0,k)}{f(k)}$ is known for all $k > 0, f(k) := f(0, k)$ and $f(x, k)$ is the Jost function corresponding to a potential $q(x) \in L_{1,1}$. The function $I(k)$ is known as the impedance function [4], and it can be measured in some problems of electromagnetic probing of the Earth. The inverse problem (IP) is: *given $I(k)$ for all $k > 0$, can one recover $q(x)$ uniquely?*

This problem was solved in [4] but in [1] and [2] a new approach to this and many other inverse problems is developed. Let us outline this new approach. Suppose that there are two potentials, $q_1(x)$ and $q_2(x)$, which generate the same data $I(k)$. Subtract from equation $(l_1 - k^2)f_1 = 0$ the equation $(l_2 - k^2)f_2 = 0$, denote $f_1 - f_2 := f, q_2 - q_1 := p(x)$, and get $(l_1 - k^2)f = pf_2$. Multiply this equation by $f_1(x, k)$, integrate over $(0, \infty)$ and then by parts. The assumption $I_1(k) = \frac{f_1'(0,k)}{f_1(k)} = \frac{f_2'(0,k)}{f_2(k)} = I_2(k)$ implies $\int_0^\infty p(x)f_1(x, k)f_2(x, k)dx = 0 \forall k > 0$. Using property C_+ one concludes $p(x) \equiv 0$, that is, $q_1(x) = q_2(x)$. This is a typical scheme for proving uniqueness theorems using property C .

References

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