MATH730 Fall 2007 - Homework 1

1. Show that $\gcd(a, bc) = 1$ iff $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$.

2. Show that if $a|c$ and $b|c$ with $\gcd(a, b) = 1$ then $ab|c$.

3. Define the least common multiple $[a, b]$ of two integers a, b to be the smallest positive integer d with $a|d$ and $b|d$. For positive integers a, b show that $[a, b] = ab/gcd(a, b)$.

4. If $\gcd(m, n) = 1$ show that for any integers a, b there is an integer x with $x \equiv a \bmod m$ and $x \equiv b \bmod n$.

5. For what values of n does $\mathbb{Z}_n^*$ have a cancelation law: $ab = 0 \Rightarrow a = 0$ or $b = 0$.

6. Show that
(i) $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$
(ii) $|(0, 1)| = |\mathbb{R}|$
(iii) $|(0, 1)| \neq |\mathbb{N}|$

7. Show that there is no mapping from a set S onto S^* the set of subsets of S.

8. Use the axioms of the integers to show that for all integers a, b, c
(i) $(-a)(-b) = ab$
(ii) $a < b$ and $c < 0$ implies that $ac > bc$.
(iii) 1, the multiplicative identity in $\mathbb{Z}$, is the smallest element of $\mathbb{N}$.

9. Determine which of the following relations $\sim$ are equivalence relations on the set S.
(i) $S = \mathbb{R}$, $a \sim b$ if $ab \in \mathbb{Q}$.
(ii) $S = M_n(\mathbb{R})$, $A \sim B$ if $A = P^{-1}BP$ for some invertible P in $M_n(\mathbb{R})$.
(iii) $S = \mathbb{N}$, $a \sim b$ if $a|b^n$ and $b|a^m$ for some n, m in $\mathbb{N}$.

10. Determine which of the following relations $\sim$ are equivalence relations on the set S. If so describe the partition of S into equivalence classes.
(i) $S = \mathbb{Z}^*$, $a \sim b$ if $ab > 0$.
(ii) $S = \mathbb{Z}$, $a \sim b$ if $a + b$ is even.
(iii) $S = \mathbb{R}$, $a \sim b$ if $|a - b| \leq 3$.