Name:

MATH 512 Intro to Modern Algebra – **Final Exam**

Friday, December 16, 2005

Check that that you have all six pages - note that the pages are double-sided

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1. (12 points) (a) G is a set closed under a binary operation $*$. What properties make $< G, * >$ a group?

(b) Prove that $S = \{(a_1, a_2) \ : \ a_1 \in \mathbb{Q}^*, \ a_2 \in \mathbb{Q}\}$ is a group with binary operation

$$(a_1, a_2) * (b_1, b_2) = (a_1 b_1, a_1 b_2 + a_2 b_1).$$

2. (12 points) Suppose that $G = \left\{ x, \frac{1}{x}, 1 - x, \frac{1}{1-x}, ... \right\}$ is the group of order six generated by the the functions $f(x) = 1/x$ and $g(x) = 1 - x$ under composition, e.g $f * g = f(g(x)) = \frac{1}{1-x}$. Find the other elements of G and complete the group table. Is G isomorphic to $\mathbb{Z}_6$ or the group of permutations S_3? Justify your choice.

$*$	x	$\frac{1}{x}$	$1-x$	$\frac{1}{1-x}$		
x	x	$\frac{1}{x}$	$1-x$	$\frac{1}{1-x}$		
$\frac{1}{x}$	$\frac{1}{x}$	x	$\frac{1}{1-x}$	$1-x$		
$1-x$	$1-x$		x			
$\frac{1}{1-x}$	$\frac{1}{1-x}$		$\frac{1}{x}$			

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3. (10 points) (a) What properties must a subset S of a ring R satisfy in order to be a subring?

(b) Prove that $T = \left\{ \begin{pmatrix} a & b \\ -3b & a \end{pmatrix} \ : \ a, b \in \mathbb{Q} \right\}$ is a subring of the ring $M_2(\mathbb{Q})$.

(c) Is T a field? Why?

4. (12 points) (a) Suppose that G is a finite group and $H \leq G$, then Lagrange's theorem states that:

(b) Suppose that G is a non-cyclic group of order 27. Show that $a^9 = e$ for all a in G.

(c) How many non-isomorphic abelian groups of order 72 are there?

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5. (10 points) Let R be a commutative ring.
(a) Define what it means for a non-zero element a of R to be a zero divisor.

(b) Prove that if ab is a zero divisor then a or b must be a zero divisor.

(c) Prove that if R has no zero divisors then it has a cancellation law.

6. (28 points) Suppose that R is a commutative ring with unity.
(a) What properties must a subset I of R satisfy in order to be an ideal?

(b) Define what it means for an ideal I of R to be prime

(c) Define what it means for an ideal I of R to be maximal.

(d) Show that $I_1 = 3\mathbb{Z} \times 7\mathbb{Z}$ is an ideal but not a prime ideal of $\mathbb{Z} \times \mathbb{Z}$.

(e) Is $I_2 = \{(a, b) \ : \ a, b \in \mathbb{Z},\ 3|ab\}$ an ideal of $\mathbb{Z} \times \mathbb{Z}$? If yes give a proof, if not show why not.

(f) Prove that the ideal $I_3 = \{f \in \mathbb{Z}[x] \ : \ f(0) = 0\}$ of $\mathbb{Z}[x]$ is prime but not maximal.

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7. (36 points) Circle True (T) or False (F).

T F (a) $\mathbb{Z}[x]$ is an integral domain.

T F (b) If F is a field then $F \times F$ is a field.

T F (c) $\mathbb{Z}_{19}$ is an integral domain.

T F (d) In $\mathbb{Q}[x]$ a non-zero prime ideal is always maximal.

T F (e) $\mathbb{Z}_{10} \times \mathbb{Z}_3$ and $\mathbb{Z}_6 \times \mathbb{Z}_5$ are isomorphic abelian groups.

T F (f) A subgroup of a cyclic group is cyclic.

T F (g) A non-cyclic group has at least one proper non-trivial cyclic subgroup.

T F (h) There are 8 units in $\mathbb{Z}_{15}$.

T F (i) The non-zero elements of $\mathbb{Z}_{31}$ form a cyclic group under multiplication.

T F (j) If $|G| = 11$ the $G \simeq \mathbb{Z}_{11}$.

T F (k) The odd permutations in S_n form a normal subgroup of S_n.

T F (l) $(\mathbb{Z} \times \mathbb{Z}) \,/\, \langle (3,1) \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}$.

T F (m) The factor ring $\mathbb{Z}_2[x]/ < x >$ has order 4.

T F (n) The operation $a * b = 3/ab$ is not associative on $\mathbb{R}^*$.

T F (o) A field does not have composite characteristic.

T F (p) $\phi_g : G \to G$ given by $\phi_g(x) = gx$ is a permutation of G for any fixed g in G.

T F (q) In the quaternions $ijk = -1$.

T F (r) $\mathbb{R}[x]/ < x^2 + 4 >$ is an integral domain.

8. (14 points) For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 5 & 8 & 7 & 9 & 10 & 2 & 6 & 1 \end{pmatrix}$:

(a) Write σ as a product of disjoint cycles.

(c) Write σ as a product of transpositions.

(d) Is σ even, odd, neither or both?

(e) What is the order of σ?

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9. (12 points) (a) If $G =< a >$ is a group of order 50 then what is the order of $\langle a^{15} \rangle$?

(b) What is the order of $(25, 6)$ in $\mathbb{Z}_{30} \times \mathbb{Z}_{20}$?

(c) What is the order of the subgroup of $\mathbb{Z}_{45}$ generated by the set $\{12, 30\}$?

10. (7 points) Suppose that $\phi : R \to R'$ is a ring homomorphism. Prove that if S' is a subring of R' then its inverse image $\phi^{-1}[S']$ is a subring of R.

11. (12 points) Suppose that F is a field.
(a) Suppose that f, g are non-zero polynomials in $F[x]$. What does the division algorithm in $F[x]$ say?

(b) Find the quotient an remainder when $2x^3 + 3x + 1$ is divided by $3x + 2$ in $\mathbb{Z}_7[x]$.

(c) Suppose that I is a non-zero ideal in $F[x]$ and g a non-zero polynomial in I of minimal degree. Prove that $I =< g >$.

12. (17 points) (a) Define what it means for $\phi : R \to R'$ to be a ring homomorphism.

(b) Verify that $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_6$ given by $\phi(x, y) = 4x + 3y \pmod 6$, is a ring homorphism.

(c) What is the kernel, $\ker(\phi)$, of the map in (b)?

(d) What is the image, $\phi[\mathbb{Z} \times \mathbb{Z}]$, of the map in (b)?

(e) What does the fundamental homomorphism theorem say in the case of the map in (b)?

13. (18 points)

$$R_1 = \mathbb{Z}_3[x], \qquad R_2 = 2\mathbb{Z} \times \mathbb{R}, \qquad R_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}.$$

(i) Which of the above rings has a unity? Give each unity.

(ii) Describe the units for the ring(s) you picked in (i)

(iii) Which rings have zero divisors? Give an example in each case.

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