Name:

MATH 512 Intro to Modern Algebra – **Final Exam**

Tuesday, December 14, 2004

Check that that you have all six pages - note that the pages are double-sided

---

1. (12 points) (a) G is a set closed under a binary operation $*$. What properties make $< G, * >$ a group?

(b) Prove that $< \mathbb{Z}, * >$ is a group with binary operation $a * b = a + b - 3$.

2. (12 points) Suppose that $G = \{I, A, B, C\}$ is the abelian group of order four generated by the matrices $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ under matrix multiplication. Find the other elements of G, give its group table and classify G according to the fundamental theorem of finitely generated abelian groups. Is G cyclic?

PG SCORE/24

3. (12 points) (a) What properties must a subset S of a ring R satisfy in order to be a subring?

(b) Prove that $T = \{a + b\sqrt{5} \ : \ a, b \in \mathbb{Q}\}$ is a subring of the ring $\mathbb{R}$.

(c) Is T a field? Why?

4. (8 points) (a) State Lagrange's theorem

(b) Prove that if $|G| = p$ is prime then G is cyclic.

5. (18 points) (a) What properties must a subset I of a ring R satisfy in order to be an ideal?

(b) Prove that $I_1 = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \ : \ a, c \in \mathbb{Z} \right\}$ is an ideal of $R_1 = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \ : \ A, C, D \in \mathbb{Z} \right\}$.

(c) Is I_1 an ideal of $M_2(\mathbb{Z})$?

(d) Prove that if I is an ideal of R and I contains a unit then $I = R$.

6. (18 points) Suppose that R is a commutative ring with unity.
(i) Define what it means for an ideal I of R to be prime.

(ii) Define what it means for an ideal I of R to be maximal.

(iii) Is the ideal $I_1 = 2\mathbb{Z} \times 5\mathbb{Z}$ prime in $\mathbb{Z} \times \mathbb{Z}$? If yes give a proof, if not show why not.

(iv) Is the ideal $I_2 = \{0\} \times \mathbb{Z}$ prime in $\mathbb{Z} \times \mathbb{Z}$? If yes give a proof, if not show why not.

(v) Is the ideal $I_2 = \{0\} \times \mathbb{Z}$ maximal in $\mathbb{Z} \times \mathbb{Z}$? If yes give a proof, if not show why not.

PG SCORE/36

7. (36 points) Circle True (T) or False (F).

T F (a) $\mathbb{Q}^*$ is a ring.

T F (b) By Euler's Little Theorem $a^{m-1} \equiv 1 \pmod{m}$ for any integers a and m with $\gcd(a, m) = 1$.

T F (c) $H \lhd G$ iff $ghg^{-1} \in G$ for all $g \in G$, $h \in H$.

T F (d) A finite field is an integral domain.

T F (e) $\mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ are isomorphic abelian groups.

T F (g) By the fundamental theorem any group of order 6 must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.

T F (h) A maximal ideal is always prime.

T F (i) If a is an element in an infinite group then a has infinite order.

T F (j) There are 4 units in the ring $\mathbb{Z}_8$.

T F (k) $\mathbb{Z}_8$ is an integral domain but not a field.

T F (l) $(\mathbb{Z} \times \mathbb{Z}) / \langle(1, 2)\rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$.

T F (m) If a group G has a cyclic subgroup then G must be abelian.

T F (n) The alternating group A_n is a normal subgroup of the symmetric group S_n.

T F (o) A simple group is one with no non-trivial proper subgroups.

T F (p) $\phi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$ is an automorphism of G for any fixed g in G.

T F (q) A non-trivial subgroup of the symmetric group S_n must contain an even permutation.

T F (r) A field must have characteristic zero.

T F (s) $\mathbb{R}[x]/ < x^2 - 2 >$ is not a field.

8. (14 points) For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 2 & 1 \end{pmatrix}$:

(a) Write σ as a product of disjoint cycles.

(c) Write σ as a product of transpositions.

(d) Is σ even, odd, neither or both?

(e) What is the order of σ?

9. (12 points) (a) What is the order of 21 in $\mathbb{Z}_{60}$?

(b) What is the order of $(6, 5)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$?

(c) What is the subgroup of $\mathbb{Z}$ generated by the set $\{45, 66\}$?

10. (20 points) Suppose that $\phi : G \to G'$ is a homomorphism from the group G onto G'.
(a) Prove that $\phi(e) = e'$ where e and e' are the respective identities of G and G'.

(b) What must be shown in order to prove that a subset H of G is a subgroup?

(c) Define $\ker(\phi)$, the kernel of ϕ.

(d) Prove that $\ker(\phi)$ is a subgroup of G.

(e) What must be shown in order to prove that a subgroup H of G is a normal subgroup?

(f) Prove that $\ker(\phi)$ is a normal subgroup of G.

PG SCORE/32

11. (17 points) (a) Define what it means for $\phi : R \to R'$ to be a ring homomorphism.

(b) Verify that $\phi : \mathbb{Z} \to \mathbb{Z}_{12}$ given by $\phi(x) = 4x \pmod{12}$ is a ring homorphism.

(c) What is the kernel, $\ker(\phi)$, of the map in (b)?

(d) What is the image, $\phi[\mathbb{Z}]$, of the map in (b)?

(e) What does the fundamental homomorphism theorem say in the case of the map in (b)?

12. (21 points)

$$R_1 = \mathbb{Z}_5[x], \qquad R_2 = \mathbb{Z}_2 \times \mathbb{Z}_4, \qquad R_3 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ : \ a, b, d \in \mathbb{Q} \right\}.$$

(i) Which of the above rings has a unity? Give each unity.

(ii) Describe the units for the ring(s) you picked in (i)

(iii) Which rings have zero divisors? Give an example in each case.

PG SCORE/38