

Name:

MATH 512 Intro to Modern Algebra – **Exam III**

Wednesday, November 30, 2005

Check that that you have all four pages - note that the pages are double-sided

1. (4 points) Suppose that R is a set closed under two binary operations $+$ and \cdot , and that $\langle R, + \rangle$ is an abelian group. What additional properties make $\langle R, +, \cdot \rangle$ into a ring?

2. (18 points)

$$R_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

(a) What properties must a subset R_1 of a ring satisfy in order to be a subring?

(b) Prove that R_1 is a subring of the ring $M_2(\mathbb{R})$.

(c) What does it mean for the rings R and R' to be isomorphic?

(d) Prove that $R_1 \simeq \mathbb{C}$.

3. (12 points) (a) Define the characteristic of a ring R .

(b) Give the characteristics of $R_1 = \mathbb{Z}_{15} \times \mathbb{Z}_{35}$ and $R_2 = \mathbb{Z} \times \mathbb{Z}_{35}$.

(c) Explain (do not just quote a theorem!) why an integral domain can not have characteristic 9.

4. (20 points) Circle True (T) or False (F).

T F (a) If $G \triangleleft H$ then $hgh^{-1} \in G$ for all $g \in G$ and $h \in H$.

T F (b) If $H, K \leq G$ with $H \simeq K$ then $G/H \simeq G/K$.

T F (c) If p is a prime then $a^{p-1} \equiv 1 \pmod{p}$ for all integers a .

T F (d) S_6 is simple.

T F (e) In the quaternions $(i + j)(i - j) = -2k$.

T F (f) If $\phi : G \rightarrow G'$ is a homomorphism then $G'/\text{Ker}\phi \simeq \phi[G]$.

T F (g) \mathbb{Z}_n is a field iff n is prime.

T F (h) $F_1 = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$ is a field.

T F (i) $F_2 = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$ is a field.

T F (j) If $c \neq 0$ is not a zero divisor and $ac = bc$ then $a = b$.

5. (6 points) Find (with justification) all the ring homomorphisms $\phi : \mathbb{Z} \rightarrow 2\mathbb{Z} \times \mathbb{Z}$.

6. (20 points)

$$R_1 = \mathbb{Q}[x], \quad R_2 = \mathbb{Z}_3 \times \mathbb{Z}_2, \quad R_3 = \left\{ \begin{pmatrix} 2a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{Z} \right\}.$$

(i) Which rings are commutative?

(ii) Which rings have a unity? Give the unity.

(iii) Describe the units for the ring(s) you picked in (ii)

(iv) Which rings have zero divisors? Give an example in each case.

(v) Which rings, if any, are integral domains? Which, if any, are fields?

7. (10 points) Give a representative from each coset and classify the group according to the fundamental theorem of finitely generated abelian groups.

(i) $(\mathbb{Z} \times \mathbb{Z}) / (3\mathbb{Z} \times \mathbb{Z})$.

(ii) $(\mathbb{Z} \times \mathbb{Z}) / \langle (3, 1) \rangle$.

8. (10 points) Suppose that N, M are normal subgroups of a group G .

(a) Prove that $NM = \{nm : n \in N, m \in M\}$ is a subgroup of G .

(b) Prove that NM is a normal subgroup of G .