

Name:

MATH 512 Intro to Modern Algebra – **Exam III**

Wednesday, December 1, 2004

Check that that you have all four pages - note that the pages are double-sided

1. (24 points) $R_1 = \left\{ \begin{pmatrix} a & b \\ 3b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}, \quad R_2 = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}.$

(a) What properties must a subset S of a ring R satisfy in order to be a subring?

(b) Prove that R_1 is a subring of the ring $M_2(\mathbb{Z})$.

(c) Is R_1 a field? Why?

(d) What does it mean for a map $\phi : R \rightarrow R'$ to be a ring homomorphism.

(e) Give a ring homomorphism $\phi : R_1 \rightarrow R_2$ which is onto. Check that it is a ring homomorphism.

(f) Is $R_1 \simeq R_2$?

2. (8 points) (a) Evaluate $\phi(33)$, the order of the multiplicative group of units in \mathbb{Z}_{33} .

(b) Use Euler's Theorem to find the remainder when 7^{1002} is divided by 33.

3. (20 points) Circle True (T) or False (F).

T F (a) If $\phi : G \rightarrow G'$ is a homomorphism then $G/\phi[G] \simeq \text{Ker}\phi$.

T F (b) If $R' \leq R$ are rings with unity then $1_R = 1_{R'}$.

T F (c) If $H \triangleleft G$ then $ghg^{-1} = h$ for all $g \in G, h \in H$.

T F (d) There is no integral domain of characteristic 6.

T F (e) $2\mathbb{Z}$ and \mathbb{Z} are isomorphic as abelian groups but not as rings.

T F (f) In the quaternions $i(i + j) = -1 + k$.

T F (g) The alternating group A_7 is simple.

T F (h) If ab is a zero divisor then a or b must be a zero divisor.

T F (i) $\mathbb{Z}_3 \times \mathbb{Z}_7$ is a field.

T F (j) A unit is never a zero divisor.

4. (8 points) Find (with justification) all the ring homomorphisms $\phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$.

5. (24 points)

$$R_1 = \mathbb{Z}_{10}$$

$$R_2 = 2\mathbb{Z} \times \mathbb{R}$$

$$R_3 = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$$

(i) Which rings are commutative?

(ii) Which rings have a unity? Give the unity.

(iii) Describe the units for the ring(s) you picked in (ii)

(iv) Which rings have zero divisors? Give an example in each case.

(v) What properties make a ring a field? Which of these rings are fields?

6. (8 points) Find the order of the factor group $(\mathbb{Z}_9 \times \mathbb{Z}_3) / \langle (3, 1) \rangle$. Give a representative from each coset and classify the group according to the fundamental theorem of finitely generated abelian groups.

7. (8 points) (a) Define what it means for a ring R to be an integral domain?

(b) Prove that a finite integral domain D every non-zero element a is a unit.