Name:

MATH 512 Intro to Modern Algebra – **Exam I**

Wednesday, September 28, 2005

Check that that you have all four pages - note that the pages are double-sided

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1. (20 points) (a) Let $< S, * >$ and $< S', *' >$ be binary algebraic structures. What properties must the function $\phi\ :\ S \rightarrow S'$ satisfy in order to be an isomorphism?

(b) Prove that if $\phi\ :\ S \rightarrow S'$ is an isomorphism for $< S, * >$ and $< S', *' >$ and $*$ is commutative on S then $*'$ is commutative on S'.

(c) Let $U_4 = \{-1, 1, i, -i\}$ denote the fourth roots of unity. Show that there can be no isomorphism ϕ between $\langle U_4, \cdot \rangle$ and $< \mathbb{Z}_4, +_4 >$ with $\phi(i) = 2$.

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2. (12 points) Suppose that $G = \{e, a, b, c, d, h, f, g\}$ is a group with identity e.

(a) Complete the group table.

$*$	e	a	b	c	d	h	f	g
e	e	a	b	c	d	h	f	g
a	a	e	d		b	f	h	
b	b	d		f	a	g	c	h
c	c	g	f				b	a
d	d	b		h		c	g	f
h	h	f	g	d	c	e		
f	f	h	c	b	g	a		
g	g		h	a	f		d	

(b) Is G abelian?

(c) How many elements in G satisfy $x * x = e$?

(d) Show that G is not isomorphic to $\langle \mathbb{Z}_8, +_8 \rangle$.

3. (20 points) Circle True (T) or False (F).

T F (a) $< \mathbb{Q}, + >$ is a group.

T F (b) $< \mathbb{Q}^*, \cdot >$ is a group.

T F (c) $< \mathbb{Z}, \cdot >$ is a group.

T F (d) Suppose that $a \sim b$ if $a + b$ is a multiple of 3, then $\sim$ is a equivalence relation on $\mathbb{Z}$.

T F (e) If $< G, * >$ is a group with $(a * b)^2 = a^2 * b^2$ for all a, b in G then G is abelian.

T F (f) The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is an identity for $H = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \ : \ a, b \in \mathbb{Q}^* \right\}$ under multiplication.

T F (g) In a group $< G, * >$ the equation $a * x = b$ has the unique solution $x = b * a^{-1}$.

T F (h) The relation $a * b = 3a + 3b$ on $\mathbb{Z}$ is commutative.

T F (i) The relation $a * b = 3a + 3b$ on $\mathbb{Z}$ is associative.

T F (j) Let $\alpha : S \to T$ and $\beta : T \to U$, be maps. If $\beta \circ \alpha$ is onto then β is onto.

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4. (20 points)

$$H = \left\{ \begin{bmatrix} 2^i & 0 \\ 0 & 2^{-i} \end{bmatrix} \ : \ i \in \mathbb{Z} \right\}$$

(a) Show that the set H is closed under matrix multiplication.

(b) Show that H contains an identity under multiplication.

(c) Show the additional property needed to prove that $< H, \cdot >$ is a subgroup of the group $< GL(2, \mathbb{Q}), \cdot >$.

(d) Show that $< H, \cdot >$ is isomorphic to $< \mathbb{Z}, + >$.

5. (8 points) Let $*$ be an associative binary operation on a set G. Let a be some element of G and define

$$H_a = \{x \in G \mid a * x = x * a\}.$$

Prove that H_a is closed under $*$.

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6. (20 points) (a) Define what it means for $< G, * >$ to be a group.

(b) Verify that the set $S = \{x \in \mathbb{R} \mid x \neq -1\}$ is closed under the binary operation $a * b = a + b + ab$.

(c) Prove that $< S, * >$ is a group

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