Name:

MATH 510 Discrete Math – **Final Exam**

Monday, December 10, 2007

Check that that you have all six pages. Show all your work and reasoning.

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1. (15 points) Find the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$ for the recurrence

$$h_n = 2h_{n-1} + 15h_{n-2}, \quad h_0 = 3, \; h_1 = -1.$$

(b) **Use this generating function** to solve the recurrence: $h_n =$ _____.

2. (10 points) The Fibonacci sequence satisfies the recurrence

$$f_n = \underline{\hspace{4cm}}, \qquad f_1 = 1, \; f_2 = 1.$$

Use induction to prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

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3. (12 points) The inclusion-exclusion principle for three sets $A, B, C \subseteq S$ states that:

$$|\overline{A} \cap \overline{B} \cap \overline{C}| =$$

Use this to count the number of integers from 1 to 1000 which are not multiples of 10, 4 or 15.

4. (12 points) Seven houses in a row are to be each painted green, yellow, blue, white or purple. How many ways if:

(a) Adjacent houses should not be the same color.

(b) At least one of the houses must be green.

(c) Exactly three houses should be blue, two green, and the remaining two not the same color.

5. (8 points) Two five member teams each with a specified team captain are to be chosen from 20 volunteers, 8 of them girls and 12 boys. One team is to be all girl, the other team is unrestricted except that the team captain must be a boy. How many different ways can the teams and captains be selected?

6. (7 points) Ten people are to be seated around a circular table. How many circular arrangements are possible if Alison, Bob, Carol and Doug all refuse to sit next to or opposite from Eric?

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7. (10 points) Use the deferred acceptance algorithm to find the **women-optimal** stable complete marriage for the preferential ranking matrix

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1,3 & 3,2 & 2,1 & 4,1 \\ 4,1 & 3,3 & 1,4 & 2,2 \\ 2,4 & 3,4 & 4,3 & 1,4 \\ 1,2 & 2,1 & 3,2 & 4,3 \end{bmatrix}$$

a marries _____
b marries _____
c marries _____
d marries _____

Here rows A, B, C, D are the women and columns a, b, c, d are the men.

8. (15 points)(a) Give the rook bipartite graph for the 6-by-6 chessboard with forbidden positions shown.

X	X			X	
	X	X			X
			X		
X	X	X		X	
X	X	X		X	
X	X	X		X	

(b) Find a max-matching. Run the matching algorithm to verify this and indicate the corresponding non-attacking rooks on the board.

(c) A min-cover is { }.

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9. (20 points) (a) How many times do you need to take your pen off the paper to trace the complete graph k_6? Explain.

(b) Does the complete bipartite graph $k_{3,4}$ have a Hamilton path? If so draw one, if not say why not.

(c) Define what it means for a (simple) graph G of order n to have the Ore property.

(d) Prove that a (simple) graph of order n with two connected components can not have the Ore property.

(e) A (simple) graph G has degree sequence $(4, 4, 4, 4, 4, 4)$. Can G have a bridge? Explain.

10. (14 points) (a) Suppose that h_n is the number of ways to choose n pieces of fruit from an unlimited choice of apples, oranges, pears, pineapples and mangos.

$$h_n = \Big(\qquad \Big).$$

(b) Suppose that in (a) the apples are loose but the oranges are in packs of 3, the pears in bags of 5, there are only four pineapples and two mangoes.

(i) Find and simplify the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$.

(ii) Give a formula for $h_n =$ _____.

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11. (12 points) Suppose that G is a tree of order 9 with a vertex of degree 6.

(a) How many edges does G have?

(b) What degree sequences can G have?

(c) Draw three non-isomorphic trees of order 9 with a vertex of degree 6.

12. (14 points) How many ways can you put 6 non-attacking rooks on the 6-by-6 chessboard with forbidden positions shown?

X	X				
	X	X			
			X	X	
					X
				X	

13. (12 points) (a) Evaluate the coefficient of $x^3y^3w^2$ in the multinomial expansion of $(2x - y + 3z + 5w)^8$.

(b) Use the binomial theorem $(1+x)^n = \sum_{j=0}^{n}$ _____ to evaluate the sum $\sum_{j=0}^{n} j\binom{n}{j}4^{j-1} =$ _____.

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14. (25 points) G is a connected general planar graph of order n with e edges and r regions.

(a) State Euler's formula relating n, e and r:

(b) Explain how to find e, the number of edges of G from the degree sequence $(d_1, ..., d_n)$.

(c) Find e, n and r for degree sequence (5,5,4).

(d) Draw five non-isomorphic, **general**, connected, planar graphs with degree sequence (5,5,4).

(e) Are the complete bipartite graphs $k_{2,3}$ or $k_{3,3}$ planar? Explain.

15. (14 points) Bob picks 11 integers from 1 to 20. When using the box principle label your boxes clearly.

(a) Prove that his choice must include integers a, b with one a multiple of the other.

(b) Prove that his choice includes three numbers differing from each other by at most 3.

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