Name:

MATH 510 Discrete Math – **Final Exam**

Tuesday, December 12, 2006

Check that that you have all five pages. Show all your work and reasoning.

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1. (15 points) Find the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$ for the recurrence

$$h_n = h_{n-1} + 6h_{n-2}, \;\; h_0 = 8, \; h_1 = 9.$$

(b) Use this generating function to solve the recurrence: $h_n =$ _____.

2. (14 points) G is a connected general planar graph of order n with degree sequence $(d_1, ..., d_n)$.
(a) Explain how you would find e, the number of edges of G. Find e for degree sequence (5,5,2).

(b) Explain how you would find r, the number of regions of G. Find r for degree sequence (5,5,2).

(c) Determine four non-isomorphic, **general**, connected, planar graphs with degree sequence (5,5,2).

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3. (7 points) Give a combinatorial explanation of the identity $\sum_{j=0}^{m} \binom{2m}{j}\binom{m}{m-j} = \binom{3m}{m}$.

4. (14 points) (a) A classroom has three rows of seats, with ten seats in each row. How many different ways can 12 students be seated if 4 always sit in the back row, 2 always sit in the front row and the remaining 6 can sit anywhere?

(b) A five character password is to contain three digits and two capital letters with no character repeated eg 7A3B5, 234DE, E178F,... but not 7A3A5, 7231B,... . How many passwords are there?

5. (7 points) Five men and five women are to be seated around a circular table so that no two men (and no two women) end up sitting together. How many circular arrangements are possible if Alison refuses to sit next to her ex Bill?

6. (7 points) Use the binomial theorem $(1+x)^{99} = \sum_{j=0}^{99} \_\_\_\_\_$ to evaluate the sum $\sum_{j=0}^{99} \binom{99}{j} \frac{(-1)^{j+1}}{j+1}$

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7. (10 points) Use the deferred acceptance algorithm to find the **men-optimal** stable complete marriage for the preferential ranking matrix

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1,3 & 2,2 & 3,1 & 4,1 \\ 4,1 & 2,3 & 1,4 & 3,2 \\ 3,4 & 2,4 & 4,3 & 1,4 \\ 1,2 & 2,1 & 3,2 & 4,3 \end{bmatrix}$$

A marries _____
B marries _____
C marries _____
D marries _____

Here rows A, B, C, D are the women and columns a, b, c, d are the men.

8. (10 points) The inclusion-exclusion principle for three sets $A, B, C \subseteq S$ states that:

$|\overline{A} \cap \overline{B} \cap \overline{C}| =$

Use this to count the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 12, \quad 0 \leq x_1, \;\; 0 \leq x_2 \leq 5, \;\; 0 \leq x_3 \leq 4, \;\; 0 \leq x_4 \leq 3.$$

9. (6 points) Evaluate the coefficient of $x^2y^3z^2w$ in the multinomial expansion of $(2x - y + 3z + 5w)^8$.

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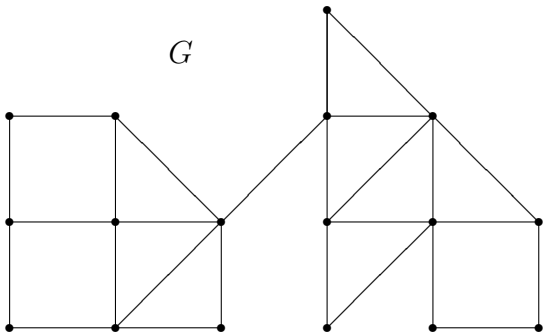
10. (12 points) (i) Does G have an open Euler trail? Explain



(ii) Does G have a Hamilton cycle? Explain.

(iii) Show that G has a Hamilton path.

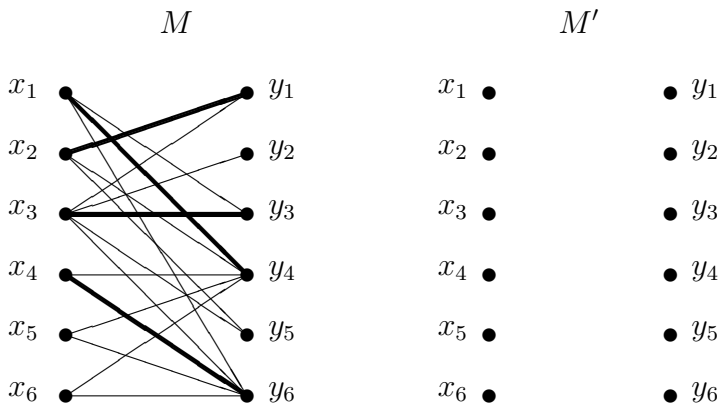
11. (10 points) Suppose that h_n is the number of ways to choose n pieces of fruit from an unlimited choice of apples, oranges, pears, bananas and mangos, where the apples, oranges and pears are loose, the bananas are in bunches of five, and you want at most four mangoes.

(a) Find and simplify the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$.

(b) Give a formula for $h_n =$ _____.

12. (10 points) For the bipartite graph and matching M:

(a) Find an M-alternating chain and hence a new matching M' with 5 edges.



(b) Show that it's a max matching by finding a cover S with 5 vertices:

$S = \{$ $\}$

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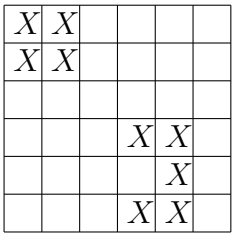
13. (12 points) Suppose that G is a tree of order n.

(a) How many edges does G have?

(b) Prove that if G has a vertex of degree 5 then it must have at least 5 pendent vertices.

(c) Draw the three non-isomorphic trees of degree 8 with a vertex of degree 5.

14. (10 points) How many ways can you put 6 non-attacking rooks on the 6-by-6 chessboard with forbidden positions shown?



15. (6 points) Bob picks 25 integers from $51, 52, ..., 97, 98$. Use the box principle to prove that he must have picked three whose digits sum up to the same amount (show clearly what boxes you are using).

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