Name:

MATH 510 Discrete Math – Exam II

Wednesday, July 21, 2004

Check that that you have all four pages. Show all your work and reasoning.

1. (18 points) Consider the nonhomogeneous recurrence relation

$$h_n = 6h_{n-1} - 9h_{n-2} + 8, \quad h_0 = 7, h_1 = 23.$$

(a) Give the general solution of the related homogeneous recurrence $h_n = 6h_{n-1} - 9h_{n-2}$.

$h_n - 6h_{n-1} + 9h_{n-2} = 0$

$x^2 - 6x + 9 = 0$ characteristic poly.

$(x-3)^2 = 0$ Double root $x=3$

$h_n = A\,3^n + Bn\,3^n$

(b) Find a particular solution to the nonhomogeneous recurrence.

Try $h_n = c$ constant

$c = 6c - 9c + 8$

$\Rightarrow 4c = 8 \Rightarrow c = 2$

$h_n = 2$

(c) Solve the nonhomogeneous recurrence.

General Solution: $h_n = A3^n + Bn3^n + 2$

$n=0$ $\quad 7 = A + 2 \Rightarrow A = 5$

$n=1$ $\quad 23 = 3A + 3B + 2 \Rightarrow 3B = 6 \Rightarrow B = 2$

$$h_n = 5\cdot 3^n + 2n\cdot 3^n + 2$$

2. (8 points) Evaluate the coefficient of $x^2y^3z^6$ in the multinomial expansion of $(2x - y + z^3)^7$?

$$\binom{7}{2\ 3\ 2}(2x)^2(-y)^3(z^3)^2 = \frac{7!}{2!\,3!\,2!}\,2^2(-1)^3\, x^2y^3z^6$$

$$= -840\, x^2y^3z^6$$

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3. (10 points) Find the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$ for the recurrence relation

$$h_n = 3h_{n-1} + 10h_{n-2}, \quad h_0 = 4, h_1 = 13.$$

$$h_n - 3h_{n-1} - 10h_{n-2} = 0$$

$$\begin{aligned}
g(x) &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \dots + h_n x^n + \dots \\
-3x g(x) & \quad -3h_0 x - 3h_1 x^2 - 3h_2 x^3 - \dots - 3h_{n-1} x^n - \dots \\
-10x^2 g(x) & \quad -10h_0 x^2 - 10 h_1 x^3 \dots - 10 h_{n-2} x^n - \dots
\end{aligned}$$

$$\begin{aligned}
(1-3x-10x^2)\, g(x) &= h_0 + (h_1 - 3h_0)x \\
&= 4 + x
\end{aligned}$$

$$g(x) = \frac{4+x}{1-3x-10x^2}$$

4. (12 points) (a) The inclusion-exclusion principle for three sets $A, B, C \subseteq S$ states that

$|\overline{A} \cap \overline{B} \cap \overline{C}| = |S| - (|A|+|B|+|C|) + (|A\cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C|$

(b) How many anagrams of PRUDENTIAL do not contain any of the words RAT, PIN or DUEL.

S = All perms. P, R, U, D, E, N, T, I, A, L $\qquad |S| = 10!$

A = perms with RAT, P, I, N, D, U, E, L $\qquad |A| = 8!$

B = perms with PIN, R, A, T, D, U, E, L $\qquad |B| = 8!$

C = perms with DUEL, P, I, N, R, A, T $\qquad |C| = 7!$

$A \cap B$ = perms with RAT, PIN, D, U, E, L $\qquad |A\cap B| = 6!$

$A \cap C$ = perms with RAT, DUEL, P, I, N $\qquad |A \cap C| = 5!$

$B \cap C$ = perms with PIN, DUEL, R, A, T $\qquad |B \cap C| = 5!$

$A \cap B \cap C$ = perms with RAT, PIN, DUEL $\qquad |A \cap B \cap C| = 3!$

$|\overline{A} \cap \overline{B} \cap \overline{C}| = 10! - (8!+8!+7!) + (6!+5!+5!) - 3! = 3\,544\,074$

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5. (12 points) Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 15, \qquad 0 \le x_1, x_2, x_3, x_4 \le 5.$$

S = solutions to $x_1+x_2+x_3+x_4=15$, $x_i \ge 0$ $\qquad |S| = \binom{15+3}{3} = \binom{18}{3}$

A_i = solutions with $x_i \ge 6$

$|A_i|$ = # solutions $y_1+y_2+y_3+y_4=9 \quad = \binom{12}{3}$

$|A_i \cap A_j|$ = " " $y_1+y_2+y_3+y_4=3 \quad = \binom{6}{3}$

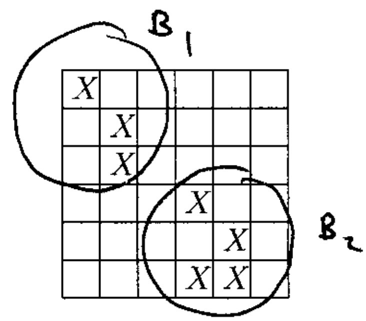
$|A_i \cap A_j \cap A_k| = 0$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - \sum|A_i| + \sum|A_i \cap A_j| - \sum|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= \binom{18}{3} - \binom{4}{1}\binom{12}{3} + \binom{4}{2}\binom{6}{3} - 0 + 0$$

$$= 56$$

6. (12 points) How many ways can you put 6 non-attacking rooks on the 6-by-6 chessboard with forbidden positions shown.



r_i = # ways to put i non-attacking rooks on forbidden squares

$r_1 = 3+4 = 7$

$r_2 = $ 1 in B_1 + 1 in B_2 + 2 in B_1 + 2 in B_2 $= 3\cdot4 + 2 + 3 = 17$

$r_3 = $ 1 in B_1 + 2 in B_2 + 2 in B_1 1 in B_2 $= 3\cdot3 + 2\cdot4 = 17$

$r_4 = $ 2 in B_1 2 in B_2 $= 2\cdot3 = 6, \qquad r_5 = r_6 = 0$

ways $= 6! - r_1 5! + r_2 4! - r_3\cdot 3! + r_4 2!$

$= 6! - 7\cdot5! + 17\cdot4! - 17\cdot3! + 6\cdot2!$

$= 198$

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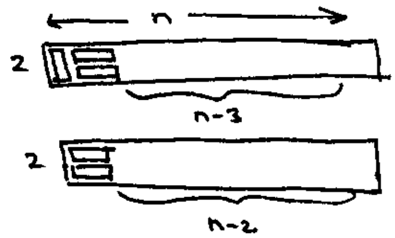
7. (8 points) Let h_n denote the number of ways in which the squares of a 2-by-n chessboard can be covered by dominoes so that no two adjacent dominoes are placed vertically. Find a recurrence relation satisfied by h_n. Do not try to solve it!



h_{n-3}

h_{n-2}

$$h_n = h_{n-2} + h_{n-3}$$

8. (12 points) Let h_n denote the number of ways to choose n pieces of fruit from a choice of (identical and unlimited) apples, peaches, oranges and pineapples, where the oranges come in packs of five and you want at most 4 pineapples.

(a) Give the generating function $g(x) = \sum_{n=0}^{\infty} h_n x^n$ and simplify.

$n = e_1 + e_2 + e_3 + e_4$ $\quad$ $e_1 = \#\text{apples}$ $\quad$ $e_3 = \#\text{oranges}$

$5 \mid e_3,\ 0 \le e_4 \le 4.$ $\quad$ $e_2 = \#\text{peaches}$ $\quad$ $e_4 = \#\text{pineapples}$

$$\sum h_n x^n = (1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)(1+x^5+x^{10}+x^{15}+\dots)(1+x+x^2+x^3+x^4)$$

$$= \frac{1}{1-x}\ \frac{1}{1-x}\ \frac{1}{1-x^5}\ \frac{1-x^5}{1-x}$$

$$= \frac{1}{(1-x)^3}$$

(b) Obtain a formula for h_n.

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n \Rightarrow h_n = \binom{n+2}{2} = \tfrac{1}{2}(n+2)(n+1)$$

9. (8 points) Five people arrive at a party wearing similar coats. How many different ways can they be handed back so that everyone leaves with the wrong coat? You can use formulae to evaluate D_n, Q_n etc. as long as you make clear what you are doing.

$$D_5 \approx 5!/e = 44.1455\dots$$

$$D_5 = 44$$

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