Name:

MATH 506 Number Theory – **Exam III**

Wednesday April 16, 2008

Check that that you have all three pages. You may assume multiplicativity where appropriate.

$\tau(n)$ = the number of positive divisors of n, $\sigma(n)$ = the sum of the positive divisors of n,

$\phi(n)$ = the Euler phi-function, $\mu(n)$ = the Möbius function, $\omega(n)$ = number prime divisors of n.

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1. (6 points) (a) Evaluate $\phi(1500) =$

(b) Evaluate $\mu(1500) =$

2. (12 points) Use the Chinese Remainder Theorem to solve the system of simultaneous congruences:

$x \equiv 3 \pmod 5$
$x \equiv 2 \pmod{12}$
$x \equiv 5 \pmod 7$

3. (12 points) (a) Find $(396, 1661)$ and write this g.c.d. as a linear combination of the two numbers.

(b) Solve the following congruence or say why no solutions exist:

$$396x \equiv 132 \ (\text{mod} 1661)$$

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4. (10 points) Define

$$F(n) = \sum_{d|n} \mu(d)^2 \tau(d)^2.$$

(a) If p is a prime then $F(p^k) =$

(b) Evaluate $F(1500) =$

5. (12 points) (a) The positive integers a and b are an amicable pair if ____________________________.

(b) $10856 = 2^3 \cdot 23 \cdot 59$ is one half of an amicable pair, find the other.

(c) Show that an even number of the form $n = 2^{m-1}(2^m - 1)$ is abundant when $2^m - 1$ is composite.

6. (14 points) Suppose that $f(n)$ is the multiplicative function satisfying

$$\mu(n)\sigma(n) = \sum_{d|n} f(d).$$

(a) From the Möbius Inversion Formula $f(n) = \sum_{d|n}$ ________________________.

(b) If p is a prime then $f(p) =$ ______________ and $f(p^2) =$ ______________. What is $f(p^k)$ for $k \geq 3$?

(c) Evaluate $f(300) =$

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7. (20 points) Circle True (T) or False (F).

T F (a) The equation $10x \equiv 15 \pmod{35}$ has 7 solutions mod 35.

T F (b) If $f(n)$ is multiplicative then $2f(n)$ is multiplicative.

T F (c) $1000x \equiv 5 \pmod{3^{1000}}$ has no solution

T F (d) $n = 15$ is deficient.

T F (e) $\sum_{d|3500} \mu(d) = 0$

T F (f) If $\sigma(n) = n + 1$ then n is prime.

T F (g) If $\mu(n) = -1$ then n is prime.

T F (h) $\#\{m \ : \ 1 \leq m \leq 1000, \ (m, 10) = 1\} = 400$.

T F (i) If $x \equiv 30 \pmod 7$ then $x \equiv 9 \pmod{21}$.

T F (j) Since $(10, 15) \neq 1$ the congruences $x \equiv 3 \pmod{10}$ and $x \equiv 8 \pmod{15}$ have no solution.

8. (8 points) (a) Define what it means for an arithmetic function $f(n)$ to be multiplicative.

(b) Prove that if $f(n)$ is multiplicative then $F(n) = \sum_{d|n} f(d)^3$ is multiplicative.

9. (6 points) Show that the following function is multiplicative: $f(n) = \begin{cases} 1 & \text{if } 3 \nmid n, \\ -1 & \text{if } 3 \mid n. \end{cases}$

Is it completely multiplicative?

PG SCORE/34