Name:

MATH 506 Number Theory – **Exam III**

Wednesday April 19, 2006

Check that that you have all three pages. Recall the arithmetic functions:

$$\tau(n) = \text{the number of positive divisors of } n, \quad \sigma(n) = \text{the sum of the positive divisors of } n,$$

$$\phi(n) = \text{the Euler phi-function}, \quad \mu(n) = \text{the Möbius function}, \quad \omega(n) = \text{number prime divisors of } n.$$

You can assume multiplicativity where appropriate.

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1. (12 points) Use the Chinese Remainder Theorem to solve the system of simultaneous congruences:

$x \equiv 8 \pmod 9$
$x \equiv 2 \pmod 8$
$x \equiv 3 \pmod 7$

2. (12 points) (a) Find $(721, 1421)$ and write this g.c.d. as a linear combination of the two numbers.

(b) Solve the following congruence or say why no solutions exist:

$$721x \equiv 35 \,(\text{mod} 1421)$$

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3. (12 points) Define
$$F(n) = \sum_{d|n} \mu^2(d)\phi(d).$$

(a) If p is a prime then $F(p^k) =$

(b) Evaluate $F(700) =$

4. (10 points) (a) A positive integer n is perfect if $\sigma(n) =$ _________.

(b) Prove that the only even perfect number of the form $n = 16m$, with m odd, is 496. Don't just quote Euler's Theorem characterizing the even perfect numbers!

5. (6 points) (a) Evaluate $\phi(700) =$

(b) Evaluate $\mu(700) =$

6. (12 points) Suppose that $f(n)$ is a function satisfying
$$3^{\omega(n)} = \sum_{d|n} f(d).$$

(a) From the Möbius Inversion Formula $f(n) = \sum_{d|n}$ ______________.

(b) If p is a prime then $f(p^k) =$

(c) Evaluate $f(700) =$

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7. (20 points) Circle True (T) or False (F).

T F (a) The equation $12x \equiv 7 \pmod{45}$ has 3 solutions mod 45.

T F (b) A reduced residue system (mod 77) contains 60 elements.

T F (c) If $\{x_1, ..., x_m\}$ is a complete residue system mod m then so is $\{ax_1, ..., ax_m\}$.

T F (d) $n = 15$ is deficient.

T F (e) $\sum_{d|1000} \phi(d) = 1000$

T F (f) $\phi(n) = n - 1$ iff n is prime.

T F (g) Wilson's Theorem says that if p is a prime then $p! \equiv -1 \pmod{p-1}$.

T F (h) If $f(n)$ is multiplicative with $f(2) = 3$ and $f(3) = 5$ then $f(12) = 45$.

T F (i) If $f(n)$ and $g(n)$ are multiplicative then $f(n)g(n)$ is multiplicative.

T F (j) The simultaneous congruences $x \equiv 7 \pmod{15}$ and $x \equiv 17 \pmod{21}$ have no solution.

8. (10 points) (a) Define what it means for an arithmetic function $f(n)$ to be multiplicative.

(b) Prove that if $f(n)$ and $g(n)$ are multiplicative then $F(n) = \sum_{d|n} f(d)g(n/d)$ is multiplicative.

9. (6 points) Show that the following function is multiplicative: $f(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Is it completely multiplicative?

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