

7. (12 points) (a) Find the prime factorization of $375 =$

(b) $\tau(n)$ is the number of positive divisors: $\tau(375) =$

(c) $\sigma(n)$ is the sum of the positive divisors: $\sigma(375) =$

(d) If $F(n) = \sum_{d|n} \tau(d)$ then $F(375) =$

8. (8 points) Use induction to prove that for all positive integers n

$$5^n \equiv 4n + 1 \pmod{16}.$$

9. (6 points) Suppose that $g(n)$ is a multiplicative function satisfying $n\tau(n) = \sum_{d|n} g(d)$.

(a) From the Möbius Inversion Formula $g(n) = \sum_{d|n} \underline{\hspace{2cm}}$.

(b) Evaluate $g(375) =$

10. (12 points) Find all right-angled triangles with coprime integer sides and one side of length 85.

11. (12 points) (a) For which odd primes p does the Legendre symbol satisfy $\left(\frac{2}{p}\right) = 1$?

(b) For which distinct odd primes p, q is $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right)$?

(c) Evaluate the Legendre symbol $\left(\frac{293}{331}\right)$

12. (24 points) Circle True (T) or False (F).

T F (a) If $15|a^2$ then $15|a$.

T F (b) If $x^2 \equiv 1 \pmod{35}$ then $x \equiv \pm 1 \pmod{35}$.

T F (c) $\{21, -3, 13, -15, -4\}$ is a complete residue system mod 5.

T F (d) $\text{ord}_{75} a \mid 20$ for all $(a, 75) = 1$.

T F (e) $7^{753} \equiv 2 \pmod{11}$.

T F (f) $\{1, 3, -3, 9\}$ is a reduced residue system mod 10.

T F (g) The Fibonacci numbers satisfy $f_{2n+3} - f_{2n+2} = f_{2n+1}$.

T F (h) $\underbrace{72727272727272727272}_{10 \text{ times}} \equiv 6 \pmod{11}$.

T F (i) If p is an odd prime then $2^p \equiv 2 \pmod{p}$.

T F (j) $6601 = 7 \cdot 23 \cdot 41$ is a Carmichael number.

T F (k) The composition $\tau(\tau(n))$ is multiplicative.

T F (l) The simultaneous congruences $x \equiv 8 \pmod{15}$ and $x \equiv -1 \pmod{18}$ have no solution.

13. (7 points) Find the continued fraction expansion of $\sqrt{7}$.

14. (10 points) (a) Use the Euclidean algorithm to compute the greatest common divisor (217,161)

(b) Solve the linear equation $217x - 161y = 21$ or explain why there are no solutions.

15. (8 points) Use the Chinese Remainder Theorem to solve the simultaneous congruences

$$x \equiv 3 \pmod{5}$$

$$x \equiv 2 \pmod{7}$$

$$x \equiv -1 \pmod{11}$$

16. (12 points) (a) Calculate the continued fraction expansion of $\frac{4169}{3864}$

(b) Calculate the continued fraction convergents

(c) A swimming contest is to be held in two pools. Contestants will swim x laps of a pool of length 38.64m or y laps of a pool of length 41.69m respectively. What would be sensible choices of whole numbers x and y so that the distance is approximately the same in both cases (assume that they want to swim less than 3km)?