Name:

MATH 506 Number Theory – **Exam III**

Wednesday April 21, 2004

Check that that you have all three pages. Recall the multiplicative functions:

$$\tau(n) = \text{the number of positive divisors of } n, \quad \sigma(n) = \text{the sum of the positive divisors of } n,$$

$$\phi(n) = \text{the Euler phi-function}, \quad \mu(n) = \text{the Möbius function}.$$

You can assume multiplicativity where appropriate.

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1. (12 points) Use the Chinese Remainder Theorem to solve the system of simultaneous congruences:

$x \equiv 3 \pmod{5}$
$x \equiv 2 \pmod{7}$
$x \equiv 1 \pmod{11}$

2. (12 points) (a) Find $(126, 497)$ and write this g.c.d. as a linear combination of the two numbers.

(b) Find all the solutions **mod 497** of the following congruence or say why no solutions exist:

$$126x \equiv 35 \pmod{497}$$

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3. (6 points) Solve the simultaneous congruences or explain why no solutions exist:

$x \equiv 7 \pmod{15}$
$x \equiv 17 \pmod{21}$

4. (9 points) (a) A positive integer n is perfect if $\sigma(n) =$ __________.

(b) Show that if $2^{k+1} - 1$ is prime then $n = 2^k(2^{k+1} - 1)$ is perfect.

5. (9 points) $6232 = 2^3 \cdot 19 \cdot 41$ is one member of an amicable pair.

(a) Evaluate $\sigma(6232) =$

(b) What is the other member of the amicable pair?

(c) Is 6232 abundant, deficient or neither?

6. (12 points) Suppose that $f(n)$ is a function satisfying

$$\tau^2(n) = \sum_{d|n} f(d).$$

(a) From the Möbius Inversion Formula $f(n) = \sum_{d|n}$ ______________.

(b) If p is a prime then $f(p^k) =$

(c) Evaluate $f(700) =$

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7. (12 points) Define

$$F(n) = \sum_{d|n} \mu(d)d^2.$$

(a) Show that $\mu(n)n^2$ is multiplicative (you may assume that $\mu(n)$ is multiplicative).

(b) If p is a prime then $F(p^k) =$

(c) Evaluate $F(700) =$

8. (20 points) Circle True (T) or False (F).

T F (a) The equation $15x \equiv 18 \pmod{35}$ has 5 solutions mod 35.

T F (b) $\{1, 5, 7, -1\}$ is a reduced residue system (mod 12).

T F (c) If $\{a, b, c, d\}$ is a reduced residue system mod 10 then so is $\{7a, 7b, 7c, 7d\}$.

T F (d) If n is odd then $\phi(2n) = \phi(n)$.

T F (e) $\sum_{d|1000} \mu(d) = 1$

T F (f) $\mu(75) = -1$.

T F (g) By Wilson's Theorem $100! \equiv 1 \pmod{101}$.

T F (h) If $n = 2^{p-1}(2^p - 1)$ is not perfect then it is abundant.

T F (i) If p is a prime then $\phi(pm) = (p-1)\phi(m)$.

T F (j) $\sum_{d|n} \phi(d) = n$

9. (8 points) For which n is $\phi(n) = 6$?

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