

(12) 5. Prove by induction that for any positive integer n the sum of the first n positive odd numbers is n^2 , that is $\sum_{k=1}^n (2k - 1) = n^2$.

(12) 6. Find the least positive integer x satisfying the congruences $x \equiv 1 \pmod{137}$ and $7x \equiv 2 \pmod{13}$. (Don't use trial and error.)

(12) 7. a) Define what it means for a function f defined on \mathbb{N} to be multiplicative.

b) Suppose that f is a multiplicative function such that $f(p) = -1$, $f(p^2) = 2$ for any odd prime p and $f(2) = 3$. Calculate the following or state that it cannot be determined.

$$f(30) =$$

$$f(18) =$$

$$f(54) =$$

(15) 8. In this problem you can leave your final answers as products of numbers. $d(n)$ is the divisor function, $\sigma(n)$ the sum of the divisors and $\phi(n)$ the Euler phi-function. Find

(a) $d(900) =$

(b) $\sigma(900) =$

(c) $\phi(900) =$

(12) 9. Prove the following theorem: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$.

(5) 10. Evaluate the Legendre symbol $\left(\frac{84}{11}\right)$

(8) 11. Use the quadratic formula to solve the congruence $x^2 + x + 1 \equiv 0 \pmod{67}$.
(Note: $8^2 \equiv -3 \pmod{67}$)

(12) 12. Given that the only solution of $x^3 + x + 1 \equiv 0 \pmod{11}$ is $x \equiv 2 \pmod{11}$, solve the congruence $x^3 + x + 1 \equiv 0 \pmod{121}$. (Don't use trial and error.)

(12) 13. Prove one of the following theorems: 1) There are infinitely many primes. 2) If n is a positive integer such that n has no prime divisor p with $p < \sqrt{n}$ then n is a prime.

Do any three of the next six problems. Additional problems will count as extra credit at a value of 4 points per problem. (You're best three will be worth 8 points each.)

- (8) 14. Define the Möbius function $\mu(n)$ and prove that it satisfies the identity $\sum_{d|n} \mu(d) = 0$ if $n > 1$.
- (8) 15. Suppose that a, b are positive integers relatively prime to 10 such that the decimal expansions of $1/a$ and $1/b$ have repeating cycles of (minimal) lengths m, n respectively. If $(a, b) = 1$ what is the length of the repeating cycle for $1/(ab)$? Prove your answer.
- (8) 16. A right triangle is to be constructed such that the base has length 100 and the other two sides have integral lengths that are relatively prime to one another. How is this possible? (Give all possibilities.)

- (8) 17. A mechanic needs a gear ratio of .84208, but each gear should have between 10 and 50 teeth. How many teeth should each gear have to best approximate this value? The continued fraction expansion of .84208 is given by

$$1 + \frac{1}{5 + \frac{1}{3 + \frac{1}{109 + \frac{1}{3}}}}$$

- (8) 18. Prove that \sqrt{p} is irrational for any prime p .

- (8) 19. Let $F_n = 2^{2^n} + 1$ be the n -th Fermat number. Suppose that there exists an integer b such that $b^{\frac{F_n-1}{2}} \equiv -1 \pmod{F_n}$. Prove that F_n is a prime.