

Math 320 Project 1.

The Fibonacci Sequence and the Golden Ratio

The Fibonacci sequence is given by 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
Let F_n denote the n -th term of the Fibonacci sequence. Thus $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, etc. The $(n - 1)$ -th and $(n + 1)$ -th terms are F_{n-1} and F_{n+1} .

- 1) In words, what is the rule for determining the terms of the Fibonacci sequence.
- 2) State the rule symbolically using the symbols F_{n-1}, F_n, F_{n+1} .
- 3) Calculate the sums of the squares of the first n terms of the Fibonacci sequence

$$\begin{aligned} S_1 &= 1^2, \\ S_2 &= 1^2 + 1^2, \\ S_3 &= 1^2 + 1^2 + 2^2, \\ S_4 &= 1^2 + 1^2 + 2^2 + 3^2, \\ S_5 &= 1^2 + 1^2 + 2^2 + 3^2 + 5^2, \text{ etc.} \end{aligned}$$

Make a chart showing the first 8 such sums S_1, \dots, S_8 . Compare the sequence S_1, \dots, S_8 with the original Fibonacci sequence (hint: consider taking products) and then make a conjecture for what S_n equals in general (in terms of the Fibonacci sequence). State your conjecture using words first, and then using symbols. Test your conjecture for $n = 9, 10$. Do you believe your conjecture is true for all values of n ?

- 4) Calculate the following:

$$\begin{aligned} P_1 &= 2 \times 1 - 1^2, \\ P_2 &= 3 \times 1 - 2^2, \\ P_3 &= 5 \times 2 - 3^2, \\ P_4 &= 8 \times 3 - 5^2. \end{aligned}$$

Continue this pattern for the next four terms. Make a conjecture and state it symbolically.

- 5) Consider the following sequence of fractions.

$$T_1 = 1, \quad T_2 = 1 + \frac{1}{1}, \quad T_3 = 1 + \frac{1}{1 + \frac{1}{1}}, \quad T_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \dots$$

Simplify the above fractions. Make a conjecture about the value of T_n (for whole numbers it might help to write $1 = \frac{1}{1}, 2 = \frac{2}{1}$), and then calculate the next two fractions T_5, T_6 to test your conjecture. Notice that

$$T_{n+1} = 1 + \frac{1}{T_n}.$$

Having calculated T_4 , what is the easiest way to calculate T_5 ?

6) A line segment is split into two pieces of lengths x , 1 with $x > 1$, such that the ratio of the longer piece to the shorter piece is the same as the ratio of the whole piece to the longer piece. Find this common ratio. It is called the **Golden Ratio** and denoted by the letter Φ . Express your answer as an exact value (using the quadratic formula) and then give a 3 decimal approximation (using a calculator).

7) Draw a rectangle of height 1 inch and length Φ inches. A rectangle with such proportions is called a **Golden Rectangle**. Divide the rectangle into two pieces, the first a square of length 1 inch and the second a smaller rectangle. Calculate the ratio of the longer side to the shorter side of the smaller rectangle. What do you find?

8) Calculate to three decimal places the ratios $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$. Do the same thing for the next six fractions in this sequence (using a calculator). What decimal value do these fractions appear to be approaching? (Can you explain why?!).