

# HEIGHTS OF ROOTS OF POLYNOMIALS WITH ODD COEFFICIENTS

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ABSTRACT. We show that the height of a non-zero non root of unity  $\alpha$  which is the zero of a polynomial with all odd coefficients of degree  $n$  satisfies

$$h(\alpha) \geq \frac{0.4278}{n+1}.$$

More generally we obtain bounds when the coefficients are all congruent to 1 modulo  $m$  for some  $m \geq 2$ .

## 1. INTRODUCTION

We recall the Mahler measure  $M(f)$  of a polynomial  $f = a \prod_{i=1}^d (x - \alpha_i)$  in  $\mathbb{C}[x]$ :

$$M(f) = |a| \prod_{i=1}^d \max\{1, |\alpha_i|\}.$$

For a non-zero algebraic number  $\alpha$  of degree  $d$  one defines the absolute logarithmic height  $h(\alpha)$  of  $\alpha$  to be

$$h(\alpha) = \frac{1}{d} \log M(F)$$

where  $F$  is an irreducible polynomial in  $\mathbb{Z}[x]$  with  $F(\alpha) = 0$ . That is  $\log M(f)$  represents the sum of the heights of the non-zero roots of  $f$  (with multiplicity) whenever  $f$  is primitive in  $\mathbb{Z}[x]$ .

For an integer  $m \geq 2$  let  $D_m$  denote the set of integer polynomials whose coefficients  $a_i$  all satisfy  $a_i \equiv 1 \pmod{m}$ . For a polynomial of degree  $n$  in  $D_m$  with no cyclotomic factors Borwein, Dobrowolski & Mossinghoff [1] proved that

$$\log M(f) \geq c_m \frac{n}{n+1}$$

with  $c_2 = \frac{1}{4} \log 5 = 0.402359\dots$ ,  $c_3 = 0.459003$  and  $c_m = \log(\sqrt{m^2+1}/2)$  for  $m > 3$ . These constants were improved in [2] to obtain  $c_2 = 0.416230\dots$ , general bounds of strength

$$c_m = \begin{cases} \log(m/2) + (3 - \log 3)/2m^2 + O(1/m^4) & \text{if } m \geq 3 \text{ odd,} \\ \log(m/2) + (4 - \log 4)/m^2 + O(1/m^4) & \text{if } m \geq 4 \text{ even,} \end{cases}$$

and particular values  $c_3 = 0.501026\dots$ ,  $c_4 = 0.832461\dots$ ,  $c_5 = 0.952869$ ,  $c_6 = 1.165884$ ,  $c_7 = 1.271775$ ,  $c_8 = 1.425369$ ,  $c_9 = 1.515669$ ,  $c_{10} = 1.634836$ ,  $c_{11} = 1.712539$ .

We show here how to more straightforwardly obtain bounds of the form

$$(1.1) \quad h(\alpha) \geq \frac{c_m}{n+1}$$

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when  $\alpha$  is a zero of a polynomial  $f$  in  $D_m$  of degree  $n$ , but not a  $2(n+1)$ st root of unity. Of course then

$$\log M(f) \geq c_m \frac{d}{n+1}$$

where  $d$  is the degree of the non-cyclotomic part of  $f$  (the type of bound obtained in Theorem 2.2 of [2]).

**Theorem 1.** *If  $\alpha$  is a zero of a polynomial  $f$  in  $D_m$  of degree  $n$  and  $\alpha$  is not an  $2(n+1)$ st root of unity (not an  $n+1$ st if  $m \geq 3$ ), then (1.1) holds with*

$$c_2 = 0.427800,$$

and

$$c_m = \log\left(\frac{m}{2}\right) + \frac{2.947486 - \frac{1}{2}\delta}{m^2} + O\left(\frac{1}{m^4}\right),$$

where

$$\delta = \begin{cases} 1 & \text{if } m \geq 3 \text{ odd,} \\ 0 & \text{if } m \geq 4 \text{ even.} \end{cases}$$

For small  $m \geq 3$  we show the following improvements:  $c_3 = 0.620362$ ,  $c_4 = 0.855600$ ,  $c_5 = 1.016628$ ,  $c_6 = 1.179916$ ,  $c_7 = 1.307083$ ,  $c_8 = 1.434141$ ,  $c_9 = 1.538934$ ,  $c_{10} = 1.640027$ ,  $c_{11} = 1.728890$ .

We note the easily obtained (if asymptotically less precise) bound

$$(1.2) \quad c_m = \begin{cases} \frac{1}{2} \log\left(\frac{m^2+3}{4}\right) & \text{if } m \geq 3 \text{ odd,} \\ \frac{1}{2} \log\left(\frac{m^2+4}{4}\right) & \text{if } m \geq 4 \text{ even,} \end{cases}$$

(the even case having already been obtained and improved in [2]). We remark that the optimal  $c_m$  in (1.1) certainly satisfies  $c_m = \log m + O(1)$ :

**Theorem 2.** *If (1.1) holds for any non root of unity  $\alpha$  that is a zero of a polynomial  $f$  in  $D_m$  of degree  $n$  and  $\alpha$ , then*

$$(1.3) \quad c_2 \leq \log\left(\frac{1+\sqrt{5}}{2}\right) = 0.481211\dots,$$

(even if we further restrict to Littlewood polynomials),

$$c_3 \leq \log 2 = 0.693147\dots,$$

$$(1.4) \quad c_4 \leq \log(1+\sqrt{2}) = 0.881373\dots,$$

$$(1.5) \quad c_6 \leq \log\left(\frac{3+\sqrt{13}}{2}\right) = 1.194763\dots$$

For general  $m \geq 3$

$$(1.6) \quad c_m \leq \log(m-1).$$

It is not clear what should be the optimal constant  $C_1$  in a bound of the form  $c_m = \log m - C_1 + o(1)$ . We note that the same computation that yields  $c_3$  immediately gives the lower bound

$$h(\alpha) \geq 0.155090$$

for abelian  $\alpha$  (see [3]).

## 2. PRELIMINARIES

Suppose that  $\alpha$  is in an algebraic number field  $k$ , and  $| \cdot |_v$ ,  $v \in V_k$  a complete set of absolute values on  $k$  normalised so that  $|x|_v = ||x||_v^{d_v/d}$  where  $d = [k : \mathbb{Q}]$ ,  $d_v = [k_v : \mathbb{Q}_v]$ , and  $||x||_v$  coincides with the usual absolute value or  $p$ -adic absolute value on  $\mathbb{Q}$ . Then

$$h(\alpha) = \log H(\alpha), \quad H(\alpha) = \prod_{v \in V_k} \max\{1, |\alpha|_v\}.$$

The normalisations ensure that this does not depend upon  $k$ .

**Lemma 1.** For  $t = 1$  or  $t > 1$  and  $k \leq 4t/(t-1)^2$

$$\sup_{|z|=1} |(z-1)^k(z+t)| = \frac{(t+1)^{k+1}}{(k+1)^{\frac{1}{2}(k+1)}} \left(\frac{k}{t}\right)^{\frac{1}{2}k}$$

achieved at  $z = -\frac{((t^2+1)k-2t)}{2t(k+1)} \pm \frac{(t+1)\sqrt{k(4t-(t-1)^2k)}}{2t(k+1)}i$ . For  $t > 1$  and  $k \geq 4t/(t-1)^2$  the supremum is  $2^k(t-1)$  achieved at  $z = -1$ .

*Proof.* Writing  $z = e^{i\theta}$ ,  $u = \cos \theta$ , it is readily checked that

$$|(z-1)^k(z+t)|^2 = 2^k(1-u)^k((t^2+1)+2tu)$$

is maximised for  $-1 \leq u \leq 1$  at  $u = -\frac{((t^2+1)k-2t)}{2t(k+1)}$  while this is at least  $-1$  (and at  $u = -1$  when  $k > 4t/(t-1)^2$ ).  $\square$

## 3. PROOF OF THEOREM 1

If  $f$  is in  $D_m$  then  $f(x) = \frac{x^{n+1}-1}{x-1} + mr(x)$  for some  $r$  of degree at most  $n$  in  $\mathbb{Z}[x]$ . Hence for  $v \nmid \infty$ , writing  $\beta = \alpha^{n+1}$ ,

$$(3.1) \quad |\beta - 1|_v = |m(\alpha - 1)r(\alpha)|_v \leq |m|_v \max\{1, |\beta|_v\}.$$

For  $m = 2$  we take

$$g(z) = (z-1)^k(z+1)^l(5z^2+6z+5)^t(29z^4+60z^3+78z^2+60z+29)^w(3z^2+2z+3)^c \\ (33x^4+60x^3+70x^2+60x+33)^e(169x^6+490x^5+871x^4+1036x^3+871x^2+490x+169)^s.$$

Thus for  $v \nmid \infty$

$$|\beta - 1|_v \leq |2|_v \max\{1, |\beta|_v\}, \quad |\beta + 1|_v = |\beta - 1 + 2|_v \leq |2|_v \max\{1, |\beta|_v\},$$

and

$$(3.2) \quad |\beta^2 - 1|_v \leq |2|_v^2 \max\{1, |\beta|_v\}^2,$$

giving

$$|\beta^2 + 1|_v = |\beta^2 - 1 + 2|_v \leq |2|_v \max\{1, |\beta|_v\}^2, \\ |5\beta^4 + 6\beta^2 + 5|_v = |5(\beta^2 - 1)^2 + 16\beta^2|_v \leq |2|_v^4 \max\{1, |\beta|_v\}^4, \\ |3\beta^4 + 2\beta^2 + 3|_v = |3(\beta^2 - 1)^2 + 8\beta^2|_v \leq |2|_v^3 \max\{1, |\beta|_v\}^4,$$

and for integer  $A, B, C, D$  (the quartic factors in  $g(z)$  correspond to  $(A, B, C) = (29, 11, 1)$  and  $(33, 12, 1)$ , and the sextic to  $(A, B, C, D) = (169, 94, 17, 1)$ ),

$$|A(\beta^2 - 1)^4 + B2^4\beta^2(\beta^2 - 1)^2 + C2^8\beta^4|_v \leq |2|_v^8 \max\{1, |\beta|_v\}^8,$$

$$|A(\beta^2 - 1)^6 + B \cdot 4^2\beta^2(\beta^2 - 1)^4 + C \cdot 4^4\beta^4(\beta^2 - 1)^2 + D4^6\beta^6|_v \leq |2|_v^{12} \max\{1, |\beta|_v\}^{12}.$$

Hence we have

$$|g(\beta^2)|_v \leq |2|_v^{2k+l+4t+8w+3c+8e+12s} \max\{1, |\beta|_v\}^{2 \deg g}, \quad v \nmid \infty.$$

For  $v \mid \infty$  and  $|\beta|_v > 1$  we observe that  $|g(\beta^2)|_v = |\beta|_v^{2 \deg g} |g(\beta^{-2})|_v$  with  $|\beta^{-2}|_v < 1$ . Hence for  $v \mid \infty$

$$|g(\beta)|_v \leq \max\{1, |\beta|_v\}^{2 \deg g} \left( \sup_{|z| \leq 1} |g(z)| \right)^{d_v/d} = \max\{1, |\beta|_v\}^{2 \deg g} \sqrt{M}^{d_v/d},$$

where, writing  $z = e^{it}$  and  $u = \cos t$ ,

$$M = \sup_{|z|=1} |g(z)|^2 = 2^{k+l+2t+4w+2c+4e+6s} L$$

with

$$L = \sup_{-1 \leq u \leq 1} (1-u)^k (1+u)^l (5u+3)^{2t} (29u^2+30u+5)^{2w} (3u+1)^{2c} (33u^2+30u+1)^{2e} (169u^3+245u^2+91u+7)^{2s}.$$

We need to justify that  $g(\beta^2) \neq 0$ . By assumption  $\beta^2 \neq 1$ , and from (3.2) plainly  $\beta^2 \neq -1$ . Observe also that

$$\begin{aligned} & 5z^{4(n+1)} + 6z^{2(n+1)} + 5, \quad 3z^{4(n+1)} + 2z^{2(n+1)} + 3, \\ & 29z^{8(n+1)} + 60z^{6(n+1)} + 78z^{4(n+1)} + 60z^{2(n+1)} + 29, \\ & 33z^{8(n+1)} + 60z^{6(n+1)} + 70z^{4(n+1)} + 60z^{2(n+1)} + 33 \end{aligned}$$

and

$$\begin{aligned} & 13z^{6(n+1)} + 2z^{5(n+1)} + 19z^{4(n+1)} - 4z^{3(n+1)} + 19z^{2(n+1)} + 2z^{(n+1)} + 13 \\ & 13z^{6(n+1)} - 2z^{5(n+1)} + 19z^{4(n+1)} + 4z^{3(n+1)} + 19z^{2(n+1)} - 2z^{(n+1)} + 13 \end{aligned}$$

(the factors of  $169z^{12(n+1)} + 490z^{10(n+1)} + 871z^{8(n+1)} + 1036z^{6(n+1)} + 871z^{4(n+1)} + 490z^{2(n+1)} + 169$ ) are all irreducible (each of their roots lies on the unit circle with the same non-trivial height so the lead coefficients of each factor would need to contain all the primes in the original lead coefficient). Since  $\alpha$  has degree at most  $n$  the remaining factors can not vanish. Thus by the product formula

$$1 = \prod_v |g(\beta^2)|_v \leq H(\beta)^{2 \deg g} 2^{-(2k+l+4t+8w+3c+8e+12s)} \sqrt{M},$$

and

$$h(\beta) \geq \frac{\log(2^{3k+l+6t+12w+4c+12e+18s}/L)}{4(k+l+2t+4w+2c+4e+6s)}.$$

The choice  $(k, l, t, w, c, e, s) = (3977, 780, 328, 96, 24, 16, 16)$  and numerical computation of  $L$  gives the lower bound  $h(\beta) \geq 0.4278003111\dots$  claimed.

For  $m = 4$  taking  $g(\beta)$  in place of  $g(\beta^2)$  immediately gives  $h(\beta) \geq 2 \cdot 0.4278003111\dots = 0.8556006223\dots$

For general  $m \geq 3$  we take

$$g(z) = \prod_{i=0}^I g_i(z)^{s_i}$$

with  $I = 2$ ,

$$g_0(z) = (z - 1),$$

$$g_1(z) = \frac{1}{2}(m - \delta)z + \frac{1}{2}(m + \delta), \quad \delta = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

and

$$g_2(z) = \frac{1}{4}(m^2 + (4 - \delta))z^2 + \frac{1}{2}(m^2 - (4 - \delta))z + \frac{1}{4}(m^2 + (4 - \delta)).$$

For  $v \nmid \infty$  we have

$$|g_1(\beta)|_v = \left| \frac{1}{2}(m - \delta)(\beta - 1) + m \right|_v \leq |m|_v \max\{1, |\beta|_v\}$$

$$|g_2(\beta)|_v = \left| \frac{1}{4}(m^2 + (4 - \delta))(\beta - 1)^2 + m^2\beta \right|_v \leq |m|_v^2 \max\{1, |\beta|_v\}^2,$$

and

$$|g(\beta)|_v \leq \max\{1, |\beta|_v\}^{\deg g} |m|_v^{\deg g}.$$

For  $v \mid \infty$  and  $|\beta|_v > 1$  writing  $|g(\beta)|_v = |\beta|_v^{\deg g} |g^*(\beta^{-1})|_v$ , where  $g^*$  is the reciprocal of  $g$ , we have

$$|g(\beta)|_v \leq \max\{1, |\beta|_v\}^{\deg g} \left( \sup_{|z| \leq 1} \max\{|g(z)|, |g^*(z)|\} \right)^{d_v/d} = \max\{1, |\beta|_v\}^{\deg g} \sup_{|z|=1} |g(z)|^{d_v/d}.$$

Hence assuming that  $g(\beta) \neq 0$  we have

$$1 = \prod_v |g(\beta)|_v \leq H(\beta)^{\deg g} m^{-\deg g} \sup_{|z|=1} |g(z)|,$$

and

$$(3.3) \quad h(\beta) \geq \log(m) - \frac{\log(\sqrt{M})}{\deg g}, \quad M := \sup_{|z|=1} |g(z)|^2.$$

It remains to check that  $g(\beta) \neq 0$ . By assumption  $\beta \neq 1$ . For  $m$  odd  $g_1(x^{n+1})$  is irreducible by Capelli's theorem (plainly  $\frac{1}{2}(m+1)/\frac{1}{2}(m-1)$  can not be a non-trivial power) so can not vanish at  $\alpha$  (which has degree at most  $n$ ). From (3.1) we know that

$$(3.4) \quad \prod_{v \mid \infty} |1 - \beta|_v \geq m \prod_{v \nmid \infty} \max\{1, |\beta|_v\}^{-1}.$$

Hence for  $m > 2$  we must have  $\beta \neq -1$  (else (3.4) gives  $2 \geq m$ ) and  $g_2(\beta) \neq 0$  (else  $\frac{1}{2}(m^2+4-\delta)(\beta-1)^2 = (4-\delta)\beta$  and (3.4) gives  $(4-\delta) = \frac{1}{4}(m^2+4-\delta) \prod_{v \mid \infty} |1-\beta|^2 \geq m^2$ ).

When  $s_2 = 0$ ,  $s_1 = 1$  (and  $s_0 \leq m^2 - 1$  when  $m$  is odd) Lemma 1 gives

$$\sqrt{M} = \frac{m^{s_0+1}}{(s_0+1)^{\frac{1}{2}(s_0+1)}} \left( \frac{4s_0}{m^2-\delta} \right)^{\frac{1}{2}s_0},$$

and

$$H(\beta)^{s_0+1} \geq (s_0+1)^{\frac{1}{2}(s_0+1)} \left( \frac{m^2-\delta}{4s_0} \right)^{\frac{1}{2}s_0}$$

The result (1.2) follows from optimally taking  $s_0 = (m^2 - \delta)/4$ .

Converting to cosines we have

$$M = \sup_{|z|=1} |g(z)|^2 = \sup_{u \in [-1, 1]} \prod_{i=0}^I f_i(u)^{s_i}$$

with

$$\begin{aligned} f_0 &= 2(1-u), \\ f_1(u) &= \frac{1}{2}(m^2-\delta)u + \frac{1}{2}(m^2+\delta), \end{aligned}$$

and

$$f_2(u) = \left( \frac{1}{2}(m^2 + (4-\delta))u + \frac{1}{2}(m^2 - (4-\delta)) \right)^2,$$

where plainly  $M$  will be achieved at  $u = -1$  or at zero of

$$\sum_{i=0}^I s_i \frac{f'_i(u)}{f_i(u)} = 0.$$

For example, after numerical computational and experimentation, the choices  $(m; s_0, s_1, s_2) = (3; 107, 48, 17), (5; 198, 26, 13), (6; 246, 21, 11), (7; 225, 14, 8), (8; 151, 7, 4), (9; 326, 12, 7), (10; 106, 3, 2), (11; 206, 5, 3)$  give us  $c_3 = 0.599206$ ,  $c_5 = 1.001086$ ,  $c_6 = 1.172140$ ,  $c_7 = 1.298988$ ,  $c_8 = 1.429512$ ,  $c_9 = 1.532875$ ,  $c_{10} = 1.637694$ ,  $c_{11} = 1.724309$ .

For the asymptotic bound we take a sequence of  $s_0, s_1, s_2$  with

$$s_0/s_2 \rightarrow Am^2, \quad s_1/s_2 \rightarrow 2C,$$

for constants  $A, C$  which we be chosen optimally below. Hence  $M$  must be achieved at

$$u = \frac{-Am^6 + m^2((4-\delta)(2C-A\delta) - 2\delta) - 2\delta(4-\delta)(C+1) \pm 2m^2\sqrt{D_1}}{(m^2-\delta)(m^2+4-\delta)(Am^2+2C+2)}$$

where

$$D_1 = m^4((2A+1+C)^2 - 8AC) + m^2((2A+1+C)(8-2\delta(C+1)) + 8AC\delta) + (4-\delta(1+C))^2,$$

or at  $u = -1$  when  $m$  is odd. Writing

$$u = -1 + \frac{2}{Am^2} \left( 2A+1+C - A\delta \pm \sqrt{D} \right) + O\left(\frac{1}{m^4}\right),$$

where  $D = (2A+1+C)^2 - 8AC$ , leads to

$$c_m \geq \log\left(\frac{m}{2}\right) + \frac{1}{2Am^2} \min_{\pm} \log\left(\frac{\exp\left(2A+1+C - A\delta \pm \sqrt{D}\right)}{\left(\frac{2A+1+C \pm \sqrt{D}}{4A}\right)^{2C} \left(\frac{-2A+1+C \pm \sqrt{D}}{4A}\right)^2}\right) + O\left(\frac{1}{m^4}\right),$$

or  $\log(m/2) + \frac{1}{Am^2} \log\left(2^{2(1+C)}/3\right) + O(m^{-4})$  if this is smaller when  $m$  is odd. For a given choice of  $C$  we can choose  $A$  to make these  $\pm$  quantities equal. Choosing (after numerical experimentation)  $2C = 1.5799148239$  and calculating  $A = 0.5569260220\dots$  gives the desired asymptotic bound.

To obtain the improved values for  $m = 3$  to  $11$  stated in the theorem we take  $g(z) = \prod_{i=0}^I g_i(z)^{s_i}$  with  $I = 4$  or  $5$  and the auxiliary factors  $g_i(z)$ , and choice of exponents  $s_i$  given in Table 1. For these  $g_j(x)$  we have  $|g_j(\beta)|_v \leq |m|_v^{\deg g_j} \max\{1, |\beta|_v\}^{\deg g_j}$  for  $v \nmid \infty$  and (3.3) holds as before (as long as  $g(\beta) \neq 0$ ). We can argue as above that  $g(\beta) \neq 0$  by irreducibility (and for  $m = 8$  that  $\frac{1}{2} \log 9 = 1.0986\dots < 1.4295\dots$  (the previous lower bound), and for  $m = 5$  and  $m = 11$  that  $\frac{1}{2} \log 8 > 1.016628$  and  $\frac{1}{2} \log 32 > 1.728890$ ).  $\square$

Additional factors could probably be added to the auxiliary polynomial  $g(x)$  in the style of [2] for further improvements.

The choices  $g(x) = (x^2 - 1)^4(x^2 + 1)$  and  $g(x) = (x - 1)^{m^2}(x + 1)$  similarly recover the values  $c_2 = \frac{1}{4} \log 5$  and  $c_m = \log(\sqrt{m^2 + 1}/2)$  for  $m > 2$  respectively (and using the auxiliary polynomials of [2] for  $g(x)$  gives the improved values stated there).

#### 4. PROOF OF THEOREM 2

Since the golden ratio is a limit point of Salem numbers with Littlewood minimal polynomials (Theorem 6.2 of [1]) we note that the optimal  $c_2$  certainly satisfies (1.3).

Suppose that  $m \geq 3$ . For (1.6) we take  $n \geq 2$  and

$$f_n(x) = x^{2n} + \sum_{i=0}^{n-1} x^{2i} - (m-1)x^{2i+1} = \frac{x^{n+1}}{x^2-1} F_n(x),$$

with

$$F_n(x) = (x^{n+1} - x^{-(n+1)}) - (m-1)(x^n - x^{-n}).$$

Since  $f_n\left(\frac{1}{m-1}\right) > 0$ ,  $f_n\left(\frac{1}{m-1}\left(1 + \left(\frac{2}{m-1}\right)^n\right)\right) < 0$  it is clear that the  $f_n(x)$  have real roots  $\alpha_n, \alpha_n^{-1}$  with  $\alpha_n \rightarrow (m-1)$  as  $n \rightarrow \infty$ . Notice that  $f_n(x)$  does not vanish at  $\pm 1$  or any  $(2n+1)$ st root of unity (so by the theorem can have no cyclotomic factors). Since  $\frac{1}{2i} F(e^{2\pi i t}) = \sin(2\pi(n+1)t) - (m-1)\sin(2\pi nt)$  changes sign it must have a zero  $t_j$  in each interval  $\left[\frac{2j-1}{4n}, \frac{2j+1}{4n}\right]$ ,  $j = 1, 2, \dots, 2n-1$  and the remaining  $(2n-2)$  zeros  $e^{2\pi i t_j}$ ,  $t_j \neq 1/2$  of  $f_n(x)$  all lie on the unit circle. Since  $f_n(x)$  has no monic factors with all roots on the unit

TABLE 1. Auxiliary factors and exponents

$m$	Auxiliary Factors $g_3(z), \dots$	$(s_0, s_1, s_2, s_3, \dots)$
3	$g_3(z) = 11(z-1)^4 + 7 \cdot 3^2 z(z-1)^2 + 3^4 z^2$ $g_4(z) = 13(z-1)^4 + 8 \cdot 3^2 z(z-1)^2 + 3^4 z^2$ $g_5(z) = 5(z-1)^2 + 2 \cdot 3^2 z$	(823, 178, 183, 48, 53, 7)
5	$g_3(z) = 8(z-1)^2 + 5^2 z$ $g_4(z) = 61(z-1)^4 + 16 \cdot 5^2 z(z-1)^2 + 5^4 z^2$ $g_5(z) = 5(11(z-1)^4 + 3 \cdot 5^2 z(z-1)^2 + 5^3 z^2)$	(340, 10, 29, 1, 8, 10)
6	$g_3(z) = 109(z-1)^4 + 21 \cdot 6^2 z(z-1)^2 + 6^4 z^2$ $g_4(z) = 11(z-1)^2 + 6^2 z$ $g_5(z) = 2(59(z-1)^4 + 11 \cdot 6^2 z(z-1)^2 + 3 \cdot 6^3 z^2)$	(222680, 19000, 8000, 2793, 2064, 1000)
7	$g_3(z) = 181(z-1)^4 + 27 \cdot 7^2 z(z-1)^2 + 7^4 z^2$ $g_4(z) = 193(z-1)^4 + 28 \cdot 7^2 z(z-1)^2 + 7^4 z^2$ $g_5(z) = 7(2z^2 + 3z + 2)$	(309, 16, 9, 4, 1, 2)
8	$g_3(z) = 2(9(z-1)^2 + 2^5 z)$ $g_4(z) = 305(z-1)^4 + 35 \cdot 8^2 z(z-1)^2 + 8^4 z^2$ $g_5(z) = 321(z-1)^4 + 36 \cdot 8^2 z(z-1)^2 + 8^4 z^2$	(944, 45, 20, 5, 5, 2)
9	$g_3(z) = 461(z-1)^4 + 43 \cdot 9^2 z(z-1)^2 + 9^4 z^2$ $g_4(z) = 481(z-1)^4 + 44 \cdot 9^2 z(z-1)^2 + 9^4 z^2$	(44277, 0, 1256, 538, 273)
10	$g_3(z) = 701(z-1)^4 + 53 \cdot 10^2 z(z-1)^2 + 10^4 z^2$ $g_4(z) = 1351(z-1)^4 + 104 \cdot 10^2 z(z-1)^2 + 2 \cdot 10^4 z^2$	(1029, 25, 10, 5, 3)
11	$g_3(z) = 32(z-1)^2 + 11^2 z$ $g_4(z) = 991(z-1)^4 + 63 \cdot 11^2 z(z-1)^2 + 11^4 z^2$ $g_5(z) = 1021(z-1)^4 + 64 \cdot 11^2 z(z-1)^2 + 11^4 z^2$	(827, 6, 12, 2, 6, 3)

circle these  $f_n(x)$  are irreducible with  $(\deg f_n + 1)h(\alpha_n) = \left(\frac{2n+1}{2n}\right) \log \alpha_n \rightarrow \log(m-1)$  as  $n \rightarrow \infty$ . For (1.4) we similarly consider

$$f_{n,4}(x) = \sum_{i=0}^{4n+2} x^i - 4x \sum_{i=0}^n x^{4i} = (1-x)(1-2x-x^2) \sum_{i=0}^n x^{4i} - x^{4n+3}$$

with real roots  $\alpha_n, \alpha_n^{-1} \rightarrow \sqrt{2}-1, \sqrt{2}+1$  and no roots at the  $(4n+3)$ st roots of unity. Writing  $F_{n,4}(x) = (x^4-1)f_{n,4}(x)x^{-(2n+3)}$  and observing that

$$\frac{1}{4i} F_{n,4}(e^{2\pi i t}) = (\cos(3\pi t) + \cos(\pi t)) \sin((4n+3)\pi t) - 2 \sin(4(n+1)\pi t)$$

has sign changes in each of the intervals  $[(2j+1)/8(n+1), (2j+3)/8(n+1)]$ ,  $j = 0, \dots, 4n+2$ , (and removing the introduced 4th roots of unity) the remaining  $4n$  zeros of  $f_{n,4}(x)$  all lie on the unit circle. For (1.5) we take

$$f_{n,6}(x) = \sum_{i=0}^{6n+4} x^i - 6x(1-x+x^2) \sum_{i=0}^n x^{6i} = (1-x)(1-x+x^2)(1-3x-x^2) \sum_{i=0}^n x^{6i} - x^{6n+5}$$

with real roots  $\alpha_n, \alpha_n^{-1} \rightarrow \frac{1}{2}(\sqrt{13} - 3), \frac{1}{2}(\sqrt{13} + 3)$  and no roots at the  $(6n + 5)$ st roots of unity. Writing  $F_{n,6}(x) = \frac{(x^6 - 1)}{(x^2 - x + 1)} f_{n,6}(x) x^{-(3n+4)}$  and observing that

$$\frac{1}{4i} F_{n,6}(e^{2\pi it}) = (\cos(3\pi t) + 2 \cos(\pi t)) \sin((6n + 5)\pi t) - 3 \sin(6(n + 1)\pi t)$$

has sign changes in each of the intervals  $[(2j + 1)/12(n + 1), (2j + 3)/12(n + 1)]$ ,  $j = 0, \dots, 6n + 4$ , (and removing the introduced 6th roots of unity) the remaining  $6n + 2$  zeros of  $f_{n,6}(x)$  all lie on the unit circle.  $\square$

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