

SHARP EXPONENTIAL INTEGRABILITY FOR TRACES OF MONOTONE SOBOLEV FUNCTIONS

PEKKA PANKKA, PIETRO POGGI-CORRADINI, AND KAI RAJALA

ABSTRACT. We answer a question posed in [12] on exponential integrability of functions of restricted n -energy. We use geometric methods to obtain a sharp exponential integrability result for boundary traces of monotone Sobolev functions defined on the unit ball.

1. INTRODUCTION

The following result answered a problem of A. Beurling, mentioned by J. Moser in a famous paper [10].

Theorem A (Chang-Marshall (1985), [1]). *There is a universal constant $C < \infty$ so that if f is analytic in \mathbb{D} , $f(0) = 0$, and*

$$(1.1) \quad \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} \leq 1,$$

then

$$\int_0^{2\pi} \exp(|f^*(e^{i\theta})|^2) d\theta \leq C,$$

where f^* is the trace of f on $\partial\mathbb{D}$, i.e., $f^*(\zeta) = \lim_{t \uparrow 1} f(t\zeta)$ for \mathcal{H}^1 -a.e. $\zeta \in \partial\mathbb{D}$.

This result is moreover “sharp” in the following sense: the Beurling functions,

$$B_a(z) := \left(\log \frac{1}{1-az} \right) \left(\log \frac{1}{1-a^2} \right)^{\frac{-1}{2}} \quad 0 < a < 1$$

are analytic in \mathbb{D} , satisfy $B_a(0) = 0$ and (1.1), and have the property that for any given $\alpha > 1$, one can choose a so that the integral

$$\int_0^{2\pi} \exp(\alpha |B_a(e^{i\theta})|^2) d\theta$$

is as large as desired.

The following is an easy corollary of the Chang-Marshall Theorem.

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Corollary A. *There is a universal constant $C < \infty$ so that if $u : \mathbb{D} \rightarrow \mathbb{R}$ is harmonic with $u(0) = 0$ and*

$$\int_{\mathbb{D}} |\nabla u(z)|^2 \frac{dA(z)}{\pi} \leq 1,$$

then

$$\int_0^{2\pi} \exp(u^*(e^{i\theta})^2) d\theta \leq C,$$

where u^* is the trace of u on $\partial\mathbb{D}$, i.e., $u^*(\zeta) = \lim_{t \uparrow 1} u(t\zeta)$ for \mathcal{H}^1 -a.e. $\zeta \in \partial\mathbb{D}$.

This can also be shown to be sharp by considering the real parts of the Beurling functions.

In [12] the last two authors generalized the Chang-Marshall theorem to quasiregular mappings in all dimensions. They also asked in [12] whether Corollary A also generalizes, perhaps substituting “harmonic” with “ n -harmonic”. In this note we show that this is indeed possible. The key concept is that of a monotone Sobolev function, whose definition we recall below, and which is quite general, and includes for instance n -harmonic functions.

2. MAIN RESULTS

For a continuous function $u : \Omega \rightarrow \mathbb{R}$, we define the oscillation of u on a compact set $K \subset \Omega$ by

$$\operatorname{osc}_K u = \max_{x,y \in K} |u(x) - u(y)|.$$

We say that $u : \Omega \rightarrow \mathbb{R}$ is *monotone* if $\operatorname{osc}_{\partial B} u = \operatorname{osc}_{\bar{B}} u$ for all n -balls B compactly contained in Ω .

Given $u : B^n \rightarrow \mathbb{R}$ in the Sobolev space $W^{1,n}(B^n)$, the radial limit

$$\tilde{u}(y) = \lim_{r \rightarrow 1} u(ry)$$

exists at \mathcal{H}^{n-1} -a.e. point $y \in S^{n-1}$. We denote by \tilde{u} the almost everywhere defined trace of u . Moreover, we denote the L^p -norm of a p -integrable $g : \Omega \rightarrow \mathbb{R}^n$ by $\|g\|_p = \|g\|_{\Omega,p}$. The surface measure $\mathcal{H}^{n-1}(S^{n-1})$ of the unit sphere S^{n-1} is ω_{n-1} . The notations $B^n(r) = B^n(0,r)$, $B^n = B^n(1)$ for n -dimensional balls will be used.

Theorem 1. *There exists a constant $C = C(n) > 0$ so that if $u \in W^{1,n}(B^n)$ is a non-constant continuous monotone function such that $u(0) = 0$, then*

$$(2.2) \quad \int_{S^{n-1}} \exp(\alpha(|\tilde{u}(y)|/\|\nabla u\|_n)^{n/(n-1)}) d\mathcal{H}^{n-1}(y) \leq C,$$

where

$$(2.3) \quad \alpha = (n-1) \left(\frac{\omega_{n-1}}{2} \right)^{\frac{1}{n-1}}.$$

The continuity assumption in Theorem 1 is of technical nature. By a theorem of Manfredi [8], so-called weakly monotone functions in $W^{1,n}$ are always continuous and monotone in the above sense. In general, $W^{1,n}$ -functions need not be continuous.

Theorem 1 is not true without the monotonicity assumption. Indeed, since the n -capacity of a point is zero, one can construct Sobolev functions $u_i \in W^{1,n}(B^n)$ so that $u_i(0) = 0$, $\|\nabla u_i\|_n \leq 1$, and $\tilde{u}_i(y) \geq i$ for every $y \in S^{n-1}$.

Our method of proof for Theorem 1 has a similar geometric flavor as in [9] and in [12], and the end-game is again to appeal to Moser's original one-dimensional proof. However, the so-called "egg-yolk" property, which was the hardest part to establish in the two papers cited above, can be quickly established in our present case. It might come as a surprise then that Theorem 1 is sharp, as we will see in Theorem 2 below, as opposed to the situation in [12].

A function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is called p -harmonic, $1 < p < \infty$, if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = 0$$

for every C^∞ -smooth test function ϕ with compact support in Ω , see [6]. Since p -harmonic functions satisfy the maximum principle ([6, 6.5]), they are, in particular, monotone.

The next result shows that the constant α in Theorem 1 is sharp.

Theorem 2. *Let α be as in Theorem 1. There exists a sequence of n -harmonic functions $u_i \in W^{1,n}(B^n)$ satisfying $\|\nabla u_i\|_n \leq 1$ and $u_i(0) = 0$, so that*

$$\int_{S^{n-1}} \exp(\beta |\tilde{u}_i(y)|^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y) \rightarrow \infty \quad \text{as } i \rightarrow \infty$$

whenever $\beta > \alpha$.

3. PROOF OF THEOREM 1

In this section we assume that u satisfies the assumptions of Theorem 1. Moreover, by considering balls $B^n(0, 1 - 1/j)$, for j large, and using Fatou's lemma, we may assume that the function u in Theorem 1 is defined in a neighborhood of the unit ball.

Lemma 3. *There exists a constant $r_0 = r_0(n) > 0$ so that if $M_0 := \max_{\bar{B}^n(r_0)} |u|$, then*

$$\int_{\{|u| \leq M_0\}} |\nabla u|^n \, dx \geq M_0^n.$$

Proof. For $0 < r < 1$ let $m := \max_{\bar{B}^n(r)} |u|$ and set $v := \min\{|u|, m\}$. By monotonicity, and since $u(0) = 0$, $\text{osc}_{S^{n-1}(t)} v = m$ for every $t \geq r$.

By the Sobolev embedding theorem on spheres, see e.g. [5, Lemma 1] or [11], there exists a constant a depending only on n such that

$$\begin{aligned} \int_{B^n \setminus \bar{B}^n(r)} |\nabla v|^n dx &= \int_r^1 \left(\int_{S^{n-1}(t)} |\nabla v|^n d\mathcal{H}^{n-1} \right) dt \\ &\geq \int_r^1 \frac{(\text{osc}_{S^{n-1}(t)} v)^n}{at} dt = \frac{m^n}{a} \log \frac{1}{r}. \end{aligned}$$

The claim follows by choosing $r_0 := \exp(-a)$. \square

Let Γ be a family of paths in a domain Ω . The n -modulus $M_n(\Gamma)$ of Γ is defined as follows:

$$M_n(\Gamma) = \inf_{\rho} \int_{\Omega} \rho^n dx,$$

where $\rho: \Omega \rightarrow [0, \infty]$ is an admissible function for Γ , i.e. a Borel function satisfying

$$(3.4) \quad \int_{\gamma} \rho ds \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. The family of all paths joining two sets $A, B \subset \bar{\Omega}$ in Ω is denoted by $\Delta(A, B; \Omega)$. We say that a given property holds for n -almost every path in a path family Γ if the property holds for all paths in $\Gamma \setminus \Gamma_0$, where Γ_0 is a subfamily of Γ having n -modulus zero.

Lemma 4. *For every $r \in (0, 1)$, there exists a constant $c = c(n, r)$, so that*

$$(3.5) \quad \mathcal{H}^{n-1}(\{y \in S^{n-1}: |u(y)| \geq s\}) \leq c \exp(-\alpha I_M^s(u))$$

for $s \geq M$. Here α is as in (2.3), $M = M(r, u) = \max_{S^{n-1}(r)} |u|$, and

$$I_M^s(u) = \int_M^s \frac{dt}{\left(\int_{\{|u|=t\}} |\nabla u|^{n-1} d\mathcal{H}^{n-1} \right)^{1/(n-1)}}.$$

Proof. Fix $r \in (0, 1)$ and $s > M = M(r, u)$. Write

$$E = E_s := \{y \in S^{n-1}: |u(y)| \geq s\}$$

and

$$U_M := \{x \in B^n: |u(x)| \geq M\}.$$

Also, here and in what follows we write

$$(3.6) \quad A_t := \int_{\{|u|=t\}} |\nabla u|^{n-1} d\mathcal{H}^{n-1}.$$

The fact that A_t is a Borel function of t is standard, see for instance [2] p. 117.

We construct an admissible function ρ for $\Delta(\partial U_M, E; B^n)$ as follows: Let $I = I_M^s(u)$, and set

$$\rho(x) := \frac{|\nabla u(x)|}{I A_{|u(x)|}^{1/(n-1)}} \chi_{U_M}(x).$$

Since every path in $\Delta(\partial U_M, E; B^n)$ has a subpath in $\Delta(\partial U_M, E; U_M)$, it suffices to show that ρ is admissible for n -almost every path in $\Delta(\partial U_M, E; U_M)$, i.e. that the set of paths where (3.4) fails has n -modulus zero. Recall that, by Fuglede's theorem [3, Theorem 3], u is absolutely continuous on n -almost every path. So, for n -almost every $\gamma \in \Delta(\partial U_M, E; U_M)$ parameterized by arc length $\ell(\gamma)$, we have, by change of variables

$$\begin{aligned} \int_{\gamma} \rho \, ds &= \int_0^{\ell(\gamma)} \frac{|\nabla u(\gamma(t))|}{I A_{|u(\gamma(t))|}^{1/(n-1)}} \, dt \geq I^{-1} \int_0^{\ell(\gamma)} \frac{|(u \circ \gamma)'(t)|}{A_{|(u \circ \gamma)(t)|}^{1/(n-1)}} \, dt \\ &\geq I^{-1} \int_M \frac{dt}{A_t^{1/(n-1)}} = 1. \end{aligned}$$

Thus ρ is an admissible function for $\Delta(\partial U_M, E; U_M)$, and so also for $\Delta(B^n(r), E; U_M)$, by the definition of n -modulus. By the coarea formula, cf. [7], we have

$$\begin{aligned} \mathbf{M}_n(\Delta(B^n(r), E; B^n)) &\leq \int_{U_M} \rho^n \, dx = I^{-n} \int_{U_M} \frac{|\nabla u(x)|^n}{A_{|u(x)|}^{n/(n-1)}} \, dx \\ &= I^{-n} \int_M \int_{\{|u|=t\}} \frac{|\nabla u(y)|^{n-1}}{A_t^{n/(n-1)}} \, d\mathcal{H}^{n-1}(y) \, dt \\ &= I^{-n} \int_M \frac{A_t}{A_t^{n/(n-1)}} \, dt = I^{1-n}. \end{aligned}$$

By the conformal invariance of n -modulus, taking inversion with respect to the unit sphere yields

$$2\mathbf{M}_n(\Delta(B^n(r), E; B^n)) \geq \mathbf{M}_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)).$$

By spherical symmetrization and [4, Theorem 4],

$$\begin{aligned} 2I^{1-n} &\geq \mathbf{M}_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)) \\ &\geq \omega_{n-1} \left(\log \frac{c_2}{\mathcal{H}^{n-1}(E)^{1/(n-1)}} \right)^{1-n}, \end{aligned}$$

where c_2 depends only on n and r . See [12] for further details. This implies (3.5). \square

Proof of Theorem 1. We will use the following result of Moser [10, Equation (6)]: If $\omega : [0, \infty) \rightarrow [0, \infty)$ is absolutely continuous and

satisfies $\omega(0) = 0$, $\omega' \geq 0$ almost everywhere, and

$$\int_0^\infty (\omega'(t))^n dt \leq 1,$$

then

$$(3.7) \quad \int_0^\infty \exp(\omega(t)^{n/(n-1)} - t) dt \leq C,$$

where $C > 0$ depends only on n . By scaling invariance of (2.2), we may assume that

$$(3.8) \quad \int_{B^n} |\nabla u|^n dx = 1.$$

Moreover, we fix $r = r_0$ and $M = M_0$ as in Lemma 3. Then, in particular, $M < 1$.

By the Cavalieri principle,

$$\begin{aligned} & \int_{S^{n-1}} \exp(\alpha|u(x)|^{n/(n-1)}) d\mathcal{H}^{n-1}(x) \\ &= \omega_{n-1} + \frac{\alpha n}{n-1} \int_0^\infty s^{1/(n-1)} \mathcal{H}^{n-1}(E_s) \exp(\alpha s^{n/(n-1)}) ds, \end{aligned}$$

where

$$E_s = \{y \in S^{n-1} : |u(y)| \geq s\}.$$

Then, by Lemma 4, it suffices to bound

$$(3.9) \quad \int_0^{\|u\|_\infty} s^{1/(n-1)} \exp(\alpha(s^{n/(n-1)} - I_M^s(u))) ds,$$

where $\|u\|_\infty = \max_{y \in S^{n-1}} |u(y)|$, and $I_M^s(u) = 0$ for $0 < s < M$. We define a function $\psi : [0, \infty) \rightarrow [0, \infty)$,

$$\psi(s) = \begin{cases} \mu s, & 0 < s < M \\ \alpha I_M^s(u) + \mu M, & M \leq s \leq \|u\|_\infty \\ \alpha I_M^{\|u\|_\infty}(u) + \mu M, & s > \|u\|_\infty, \end{cases}$$

where

$$(3.10) \quad \mu = \alpha \left(\frac{M}{\int_{\{|u| \leq M\}} |\nabla u|^n dx} \right)^{1/(n-1)}.$$

Then, by Lemma 3, $\mu M \leq \alpha$, and thus we may consider

$$(3.11) \quad \int_0^{\|u\|_\infty} s^{1/(n-1)} \exp(\alpha s^{n/(n-1)} - \psi(s)) ds$$

instead of (3.9). We define ϕ by $\phi(y) = \psi^{-1}(y)$ for $0 < y < \|\psi\|_\infty$, and $\phi(y) = \|u\|_\infty$ for $y \geq \|\psi\|_\infty$. Then, changing variables $y = \psi(s)$ in (3.11) yields

$$(3.12) \quad \int_0^\infty \exp(\alpha \phi(y)^{n/(n-1)} - y) \phi'(y) \phi(y)^{1/(n-1)} dy.$$

Integrating by parts, we then have that (3.12) equals $C_1(n) + C_2(n)T$,

$$T = \int_0^\infty \exp((\alpha^{(n-1)/n} \phi(y))^{n/(n-1)} - y) dy.$$

Now, since ϕ is absolutely continuous and increasing, and $\phi(0) = 0$, Theorem 1 follows from Moser's result (3.7) if we can show that

$$(3.13) \quad \int_0^\infty (\alpha^{(n-1)/n} \phi'(y))^n dy \leq 1.$$

We have

$$\alpha^{(n-1)/n} \phi'(y) = \begin{cases} \alpha^{(n-1)/n} \mu^{-1}, & 0 < y < \mu M \\ \alpha^{-1/n} A_{\phi(y)}^{1/(n-1)}, & \mu M < y < \|\psi\|_\infty \\ 0, & y > \|\psi\|_\infty, \end{cases}$$

where $A_{\phi(y)}$ as in (3.6). Hence,

$$(3.14) \quad \alpha^{n-1} \int_0^\infty \phi'(y)^n dy = \alpha^{n-1} \mu^{1-n} M + \alpha^{-1} \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)} dy.$$

By our choice of μ , the first term equals $\int_{\{|u| \leq M\}} |\nabla u|^n dx$. Also, by changing variables $\phi(y) = s$ in the right hand integral, and using the coarea formula, we have

$$(3.15) \quad \begin{aligned} \alpha^{-1} \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)} dy &= \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)} \phi'(y) dy \\ &= \int_M^{\|\psi\|_\infty} A_s ds = \int_{\{|u| \geq M\}} |\nabla u|^n dx. \end{aligned}$$

Combining (3.14), (3.15), (3.10) and (3.8) then yields (3.13). The proof is complete. \square

4. PROOF OF THEOREM 2

Fix $\beta > \alpha$. For notational convenience, we consider first functions in $B^n(e_n, 1)$ instead of B^n . Fix $2 \leq i \in \mathbb{N}$, and denote $\varepsilon = \varepsilon_i = i^{-1}$. Define $v = v_i: B^n(-\varepsilon e_n, 2 + \varepsilon) \rightarrow \mathbb{R}$,

$$v(x) = -\log |x + \varepsilon e_n|.$$

Then v is n -harmonic in $B^n(-\varepsilon e_n, 2 + \varepsilon) \setminus \{-\varepsilon e_n\}$. We first show that

$$(4.16) \quad \int_{B^n(e_n, 1)} |\nabla v|^n dx \leq \frac{\omega_{n-1}}{2} \log \frac{2 + \varepsilon}{\varepsilon}.$$

Clearly,

$$\int_{B^n(e_n, 1)} |\nabla v|^n dx \leq \frac{1}{2} \int_A |\nabla v|^n dx,$$

where

$$A = B^n(-\varepsilon e_n, 2 + \varepsilon) \setminus \bar{B}^n(-\varepsilon e_n, \varepsilon).$$

Since

$$|\nabla v(x)|^n = |x + \varepsilon e_n|^{-n},$$

we have

$$\frac{1}{2} \int_A |\nabla v|^n dx = \frac{1}{2} \int_{B^n(0, 2+\varepsilon) \setminus \bar{B}^n(0, \varepsilon)} |x|^{-n} dx = \frac{\omega_{n-1}}{2} \log \frac{2+\varepsilon}{\varepsilon}.$$

Hence (4.16) holds.

To study exponential integrability of v , set

$$(4.17) \quad \gamma = \beta \left(\frac{\omega_{n-1}}{2} \log \frac{2+\varepsilon}{\varepsilon} \right)^{1/(1-n)}$$

and $\tau = \gamma/(n-1)$.

By the choice of γ , (4.16), and the Cavalieri principle,

$$(4.18) \quad \begin{aligned} & \int_{S^{n-1}(e_n, 1)} \exp(\beta (|v|/\|\nabla v\|_n)^{n/(n-1)}) d\mathcal{H}^{n-1} \\ & \geq \omega_{n-1} + \frac{n\gamma}{n-1} \int_0^\infty \mathcal{H}^{n-1}(E_s) s^{1/(n-1)} \exp(\gamma s^{n/(n-1)}) ds, \end{aligned}$$

where

$$E_s = \{x \in S^{n-1}(e_n, 1) : |v(x)| \geq s\}.$$

Since

$$\begin{aligned} E_s &= S^{n-1}(e_n, 1) \cap \bar{B}^n(-\varepsilon e_n, \exp(-s)) \\ &\quad \cup S^{n-1}(e_n, 1) \setminus B^n(-\varepsilon e_n, \exp(s)), \end{aligned}$$

we have

$$(4.19) \quad \mathcal{H}^{n-1}(E_s) \geq C(n)(\exp(-s))^{n-1} = C(n) \exp((1-n)s)$$

for $0 \leq s \leq \log(1/(2\varepsilon))$.

Combining (4.18) and (4.19) yields

$$\begin{aligned} & \frac{1}{C(n)} \int_{S^{n-1}(e_n, 1)} \exp(\gamma |v|^{n/(n-1)}) d\mathcal{H}^{n-1} \\ & \geq \frac{n\gamma}{n-1} \int_0^{\log(1/(2\varepsilon))} s^{1/(n-1)} \exp(\gamma s^{n/(n-1)} + (1-n)s) ds \\ & = n\tau \int_0^{\log(1/(2\varepsilon))} s^{1/(n-1)} \exp((n-1)(\tau s^{n/(n-1)} - s)) ds \\ & = \int_0^{\log(1/(2\varepsilon))} (n\tau s^{1/(n-1)} - (n-1)) \exp((n-1)(\tau s^{n/(n-1)} - s)) ds \\ & \quad + (n-1) \int_0^{\log(1/(2\varepsilon))} \exp((n-1)(\tau s^{n/(n-1)} - s)) ds \\ & \geq \exp\left((n-1) \left(\tau \left(\log \frac{1}{2\varepsilon} \right)^{n/(n-1)} - \log \frac{1}{2\varepsilon} \right)\right) - 1. \end{aligned}$$

Since

$$\left(\log \frac{2+\varepsilon}{\varepsilon}\right)^{1/(1-n)} \left(\log \frac{1}{2\varepsilon}\right)^{1/(n-1)} \geq 1 - \delta(\varepsilon),$$

where $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have

$$(4.20) \quad \int_{S^{n-1}(e_n,1)} \exp(\gamma|v|^{n/(n-1)}) \, d\mathcal{H}^{n-1} \geq C(n)\varepsilon^{-T} - C(n),$$

where

$$T = (\beta - \alpha)(2/\omega_{n-1})^{1/(n-1)} - \delta'(\varepsilon),$$

and $\delta'(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$.

To prove Theorem 2, we consider the sequence $u_i: \bar{B}^n \rightarrow \mathbb{R}$,

$$u_i(x) = v_i(x + e_n) - v_i(e_n),$$

where $v_i(e_n) = -\log(1 - \varepsilon_i) \leq \log 2$ for all i . We fix M such that

$$\beta' = \beta \left(\frac{M - \log 2}{M}\right)^{n/(n-1)} > \alpha.$$

Set also $E_i = \{y \in S^{n-1}(e_n, 1) : |v_i(y)| \geq M\}$. Then

$$\beta|v_i(y) - v_i(e_n)|^{n/(n-1)} \geq \beta'|v_i(y)|^{n/(n-1)}$$

on E_i for every i . Thus

$$\begin{aligned} & \int_{S^{n-1}} \exp(\beta(|u_i|/\|\nabla u_i\|_n)^{n/(n-1)}) \, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}(e_n,1)} \exp(\beta(|v_i(y) - v_i(e_n)|/\|\nabla v_i\|_n)^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y) \\ &\geq \int_{E_i} \exp(\beta'(|v_i(y)|/\|\nabla v_i\|_n)^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y) \\ &\geq \int_{S^{n-1}(e_n,1)} \exp(\beta'(|v_i(y)|/\|\nabla v_i\|_n)^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y) \\ &\quad - \omega_{n-1} \exp(\beta'(M/\|\nabla v_i\|_n)^{n/(n-1)}). \end{aligned}$$

Since $\beta' > \alpha$ and $\varepsilon_i = i^{-1}$ in (4.20), the claim now follows from (4.20).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48104, USA

E-mail address: `pankka@umich.edu`

DEPARTMENT OF MATHEMATICS, CARDWELL HALL, KANSAS STATE UNIVERSITY, MANHATTAN, KS 66506, USA

E-mail address: `pietro@math.ksu.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, P. O. Box 35 (MAD), FI-40014 UNIV. OF JYVÄSKYLÄ, FINLAND.

E-mail address: `kirajala@maths.jyu.fi`