THE p-HARMONIC TRANSFORM BEYOND ITS NATURAL DOMAIN OF DEFINITION, INTERPOLATION AND CONTINUITY.

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Abstract. To every vector field $f \in \mathbf{L}^q(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, there corresponds a unique solution $u \in \mathbf{W}_0^{1,p}(\Omega)$ of the p-harmonic equation

$$\operatorname{div} |\nabla u|^{p-2}\, \nabla u \;=\; \operatorname{div} f\,,$$

where the exponents satisfy Hölder's relation: $p\,, q > 1$ and $p + q = p \cdot q$. The p-harmonic transform assigns to f the gradient of the solution.

$$\mathcal{H}_p : \mathbf{L}^q(\Omega, \mathbb{R}^n) \longrightarrow \mathbf{L}^p(\Omega, \mathbb{R}^n)\,, \qquad \mathcal{H}_p\,(f) \;=\; \nabla u$$

More general PDEs and the corresponding nonlinear transforms are also considered.
We are concerned with the continuous extension of the p-harmonic transform beyond this so-called natural domain of definition. Namely,

$$\mathcal{H}_p : \mathbf{L}^{\lambda q}(\Omega, \mathbb{R}^n) \longrightarrow \mathbf{L}^{\lambda p}(\Omega, \mathbb{R}^n)\,, \quad \text{for some parameters } \lambda \geq \max\left\{\frac{1}{p}\,,\, \frac{1}{q}\right\}$$

First, we establish an *Interpolation Theorem*. Because of nonlinearity, this result requires substantial innovations of the familiar Marcinkiewicz ideas from the linear theory. Then we explicitly identify the so-called *critical parameter* λ for which the existence, uniqueness and continuity of $\mathcal{H}_p$ take place. Surprisingly, the uniqueness property in unbounded domains is lost when λ exceeds the critical parameter. It is a little more surprising that the n-harmonic transform in unbounded domains, say $\Omega = \mathbb{R}^n$, cannot be extended to any Lebesgue space $\mathbf{L}^s(\mathbb{R}^n, \mathbb{R}^n)$, with $s > n$. In other words, the critical parameter is equal to 1 in this case.