Gerald Hart
Anton Lyubinin

Windmills. An old rusty farm windmill, the kind with fan blades that are perpendicular to the directional fin (See Figure 1), is completely still on a very calm but stormy day. A bird flies over the windmill and poops on one blade of the fan at the tip directly above the center of the fan. The windmill is still until time t=0, when a tornado comes and begins to accelerate the windmill so that the fan blades are accelerating and the fan altogether is accelerated in a circular motion due to the spinning motion of the tornado acting on the directional fin. The blades are five feet long, so when spinning, form circle with radius of five. Think of the fan as a tangent line to the circumference of the circle. This circumference has a radius of two feet. The windmill has a height of 50 feet (the distance from the base to the center of the fan. At time t=0, the poop is at (2,0,55). The tornado produces winds at 100 ft/sec. The angular acceleration of the fan blades is directly proportional to the wind velocity (With $k= \pi/600\ sec^{-1}$). The angular acceleration of the spin produced by the directional fin is also directly proportional to wind velocity (with $k=\pi/2400 sec^{-1}$). The fan blades are spinning clockwise and the spin produced by the directional fin is counter-clockwise.

a) Write parametric equations for the poop (x(t),y(t),z(t)).

b) The original maximum spin rating for this windmill is 8/3 rev/sec for the fan. Because this is an old windmill, it fails at 75% of the original maximums. Find the time t when the windmill fails.

c) Find the position of the poop at this time t.

At the time found in part (b), the windmill breaks at the base and falls in the direction of the positive x-axis. At this time t, the acceleration of the fan and the directional fin ceases and these motions are a constant velocity until it hits the ground. The windmill falls with an acceleration of $\pi/4$. (Think of the path of the fall as a circle with a radius that is the height of the windmill, and the acceleration being the angular acceleration of the angle formed by the difference in the windmill's original position and the position of the windmill during the fall at time t). (Hint: Try representing 3 motions with 3 vectors and summing them)

d) Write parametric equations for the poop (x(t),y(t),z(t)).

e) Find the time t when the windmill hits the ground.

f) Find the position of the poop at this time t.

When the windmill hits the ground, the fan blade with the poop breaks off and becomes a projectile with an initial velocity of $50\sqrt{2}$ ft/sec and leaves the ground at a 45-degree angle. The tornado is gone now, so assume air resistance to be –k(v), where $k = 0.1\ sec^{-1}$. The blade is spinning around its center of mass (assume mass to be evenly distributed) at 2 rev/sec.

g) Write parametric equations for the poop (x(t),y(t),z(t)).

h) Find the maximum height of the projectile.

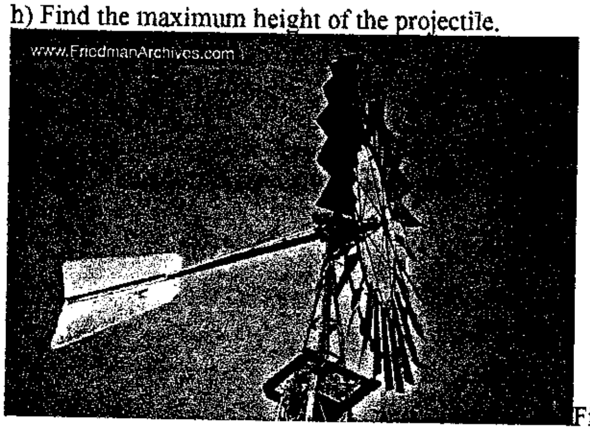


Figure 1

a) $\omega_F = K V_w \qquad K_F = \frac{\pi}{600}\,\frac{1}{sec} \qquad V_w = 100\,ft/sec$

$\omega_F = (100)(\pi/600) \qquad \omega_F = \frac{\pi}{6}\ ft/sec^2 = \frac{d^2\alpha}{dt^2}$

$\frac{d\alpha}{dt} = \frac{\pi t}{6} + C_1 \qquad \frac{d\alpha}{dt}(0) = 0 = C_1 \qquad \frac{d\alpha}{dt} = \frac{\pi t}{6}$

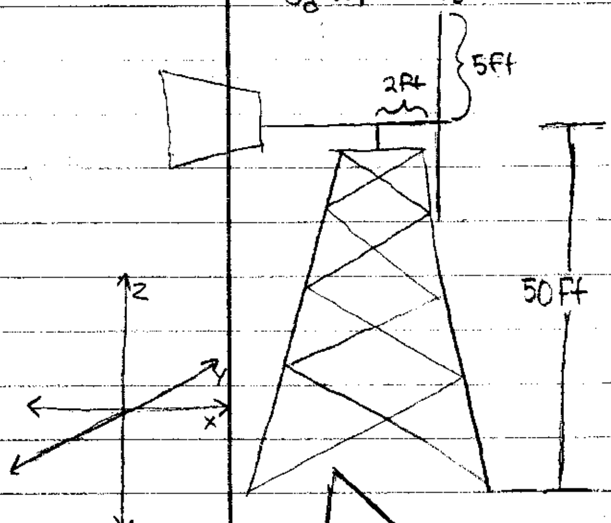
$\alpha_F(t) = \frac{\pi t^2}{12} + C_2 \qquad \alpha_F(0) = 0 = C_2 \qquad \alpha_F(t) = \frac{\pi t^2}{12}$

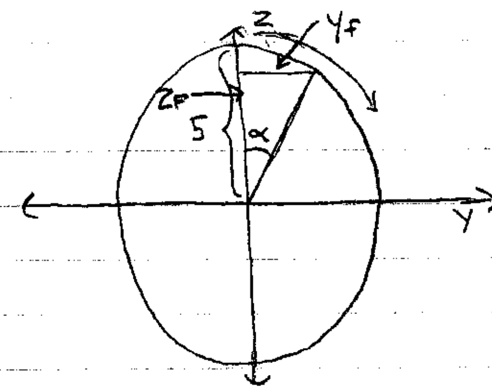
$\omega_d = K V_w \qquad K_d = \frac{\pi}{2400}\,\frac{1}{sec} \qquad V_w = 100\,ft/sec$

$\omega_d = 100(\pi/2400) \qquad \omega_d = \frac{\pi}{24}\ ft/sec^2 = \frac{d^2\theta}{dt^2}$

$\frac{d\theta}{dt} = \frac{\pi t}{24} + C_3 \qquad \frac{d\theta}{dt}(0) = 0 = C_3 \qquad \frac{d\theta}{dt} = \frac{\pi t}{24}$

$\theta_d(t) = \frac{\pi t^2}{48} + C_4 \qquad \theta_d(0) = 0 = C_4 \qquad \theta_d(t) = \frac{\pi t^2}{48}$

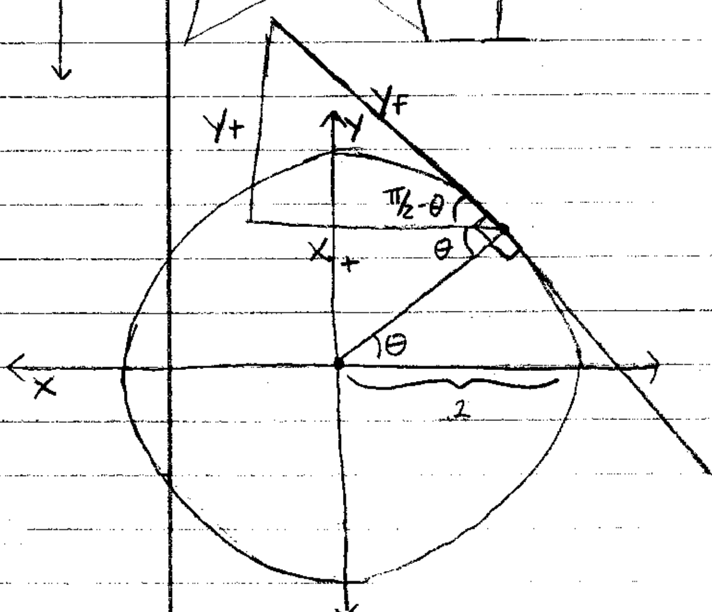




$x_F = 2$

$y_F = 5\sin\alpha$

$z_F = 5\cos\alpha$



$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{x_t}{y_F}$

$\sin\theta = \frac{x_t}{5\sin\alpha}$

$x_t = \sin\theta\, 5\sin\alpha$

$x_d = 2\cos\theta$

$x_p = x_d - x_t$

$x_p = 2\cos\theta - \sin\theta\, 5\sin\alpha$

$\boxed{x_p(t) = 2\cos\left(\frac{\pi t^2}{48}\right) - \sin\left(\frac{\pi t^2}{48}\right) 5\sin\left(\frac{\pi t^2}{12}\right.}$

$\sin\left(\frac{\pi}{2}-\theta\right) = \frac{y_t}{y_F}$

$\cos\theta = \frac{y_t}{5\sin\alpha}$

$y_t = \cos\theta \, 5\sin\alpha$

$y_d = 2\sin\theta$

$y_p = y_d + y_t$

$y_p = 2\sin\theta + \cos\theta \, 5\sin\alpha$

$$\boxed{y_p(t) = 2\sin\left(\frac{\pi t^2}{48}\right) + \cos\left(\frac{\pi t^2}{48}\right) 5\sin\left(\frac{\pi t^2}{12}\right)}$$

$z_p = z_F + 50$

$z_p = 5\cos\alpha + 50$

$$\boxed{z_p(t) = 5\cos\left(\frac{\pi t^2}{12}\right) + 50}$$

b) $\left(\frac{8}{3}\right)\cancel{(0.75)} = S_{\pi} = \left(\frac{8}{3}\right)\left(\frac{3}{4}\right) = \frac{2 \text{ rev}}{1 \text{ sec}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} = \frac{4\pi \text{ rad}}{\text{sec}} = \frac{d\alpha}{dt}$

$\frac{d\alpha}{dt} = \frac{\pi t}{6} \qquad 4\pi = \frac{\pi t}{6} \qquad \boxed{t = 24}$

c) $x_p(24) = 2\cos\left(\frac{\pi(24)^2}{48}\right) - \sin\left(\frac{\pi(24)^2}{48}\right) 5\sin\left(\frac{\pi(24)^2}{12}\right)$

$= 2\cos(12\pi) - \sin(12\pi)\, 5\sin(48\pi)$

$= 2 - 0$

$x_p(24) = 2$

$y_p(24) = 2\sin\left(\frac{\pi(24)^2}{48}\right) + \cos\left(\frac{\pi(24)^2}{48}\right) 5\sin\left(\frac{\pi(24)^2}{12}\right)$

$= 2\sin(12\pi) + \cos(12\pi)\, 5\sin(48\pi)$

$= 0 + 0$

$y_p(24) = 0$

$z_p(24) = 5\cos\left(\frac{\pi(24)^2}{12}\right) + 50$

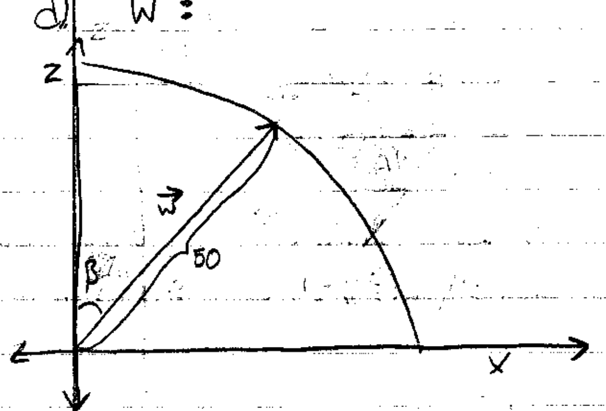
$= 5\cos(48\pi) + 50$

$= 5 + 50$

$z_p(24) = 55$

$\boxed{\text{poop @ } t=24 \quad (2, 0, 55) \quad (\text{A.K.A. original position})}$

d) $\vec{W}$:



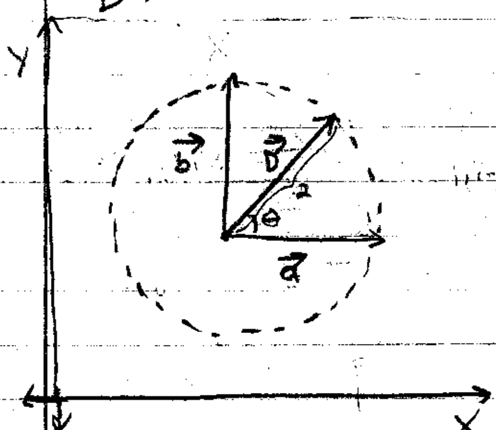
$$\frac{d^2\beta}{dt^2} = \frac{\pi}{4} \qquad \frac{d\beta}{dt} = \frac{\pi t}{4} + C_5$$

$$\frac{d\beta}{dt}(0) = 0 = C_5 \qquad \frac{d\beta}{dt} = \frac{\pi t}{4}$$

$$\beta(t) = \frac{\pi t^2}{8} + C_6 \qquad \beta(0) = 0 = C_6$$

$$\beta(t) = \frac{\pi t^2}{8}$$

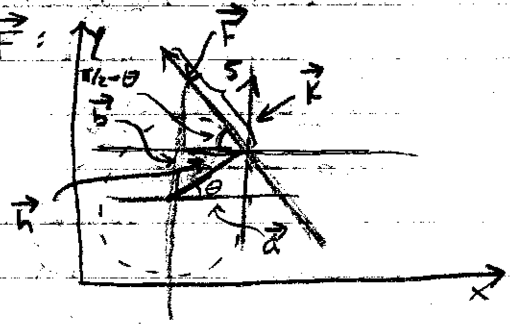
$\vec{D}$:



Stop Acceleration (Now constant Velocity) @ t=24

for $t > 24$: $\theta = \frac{\pi t}{4}$ $\quad \alpha = \pi t$

$\vec{F}$:



$$\vec{W}: \begin{pmatrix} 50(\sin\beta) \\ 0 \\ 50(\cos\beta) \end{pmatrix} \qquad \vec{a} = \begin{pmatrix} \sin\beta \\ 0 \\ -\cos\beta \end{pmatrix} \qquad \vec{b} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{D}: 2[(\cos\theta)\vec{a} + (\sin\theta)\vec{b}]$$

$$\vec{D}: \begin{pmatrix} 2(\cos\theta)(\sin\beta) \\ 2(\sin\theta) \\ -2(\cos\theta)(\cos\beta) \end{pmatrix} \qquad \vec{h} = -(\sin\theta)\vec{a} + (\cos\theta)\vec{b} \qquad \vec{h} = \begin{pmatrix} -(\sin\theta)(\sin\beta) \\ \cos\theta \\ (\sin\theta)(\cos\beta) \end{pmatrix}$$

$$\vec{k} = \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} \qquad \vec{F}: 5[(\cos\alpha)\vec{k} + (\sin\alpha)\vec{h}] \qquad \vec{F}: \begin{pmatrix} 5[(\cos\alpha)(\sin\beta) - (\sin\alpha)(\sin\theta)(\sin\beta)] \\ 5(\sin\alpha)(\cos\theta) \\ 5[(\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\theta)(\cos\beta)] \end{pmatrix}$$

$$\vec{W} + \vec{D} + \vec{F} = \begin{pmatrix} 50(\sin\beta) + 2(\cos\theta)(\sin\beta) + 5[(\cos\alpha)(\sin\beta) - (\sin\alpha)(\sin\theta)(\sin\beta)] \\ 2(\sin\theta) + 5(\sin\alpha)(\cos\theta) \\ 50(\cos\beta) - 2(\cos\theta)(\cos\beta) + 5[(\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\theta)(\cos\beta)] \end{pmatrix}$$

$$x_p(t) = 50(\sin(\tfrac{\pi t^2}{8})) + 2(\cos(\tfrac{\pi t}{4}))(\sin(\tfrac{\pi t^2}{8})) + 5(\cos(\pi t))(\sin(\tfrac{\pi t^2}{8})) - 5(\sin(\pi t))(\sin(\tfrac{\pi t}{4}))(\sin(\tfrac{\pi t^2}{8}))$$

$$y_p(t) = 2(\sin(\tfrac{\pi t}{4})) + 5(\sin(\pi t))(\cos(\tfrac{\pi t}{4}))$$

$$z_p(t) = 50(\cos(\tfrac{\pi t^2}{8})) - 2(\cos(\tfrac{\pi t}{4}))(\cos(\tfrac{\pi t^2}{8})) + 5(\cos(\pi t))(\cos(\tfrac{\pi t^2}{8})) + 5(\sin(\pi t))(\sin(\tfrac{\pi t}{4}))(\cos(\tfrac{\pi t^2}{8}))$$

e) Windmill will hit the ground when $\beta = 90°$ or $\pi/2$

$\beta = \frac{\pi t^2}{8}$ $\quad \frac{\pi}{2} = \frac{\pi t^2}{8} \quad = 2t^2 = 8 \quad t^2 = 4 \quad t = 2$

$\boxed{\text{windmill hit the ground @ } t = 2}$

f) $x_p(2) = 50(\sin(\frac{\pi(2)^2}{8})) + 2(\cos(\frac{\pi(2)}{4}))(\sin(\frac{\pi(2)^2}{8})) + 5(\cos(\pi(2)))(\sin(\frac{\pi(2)^2}{8})) - 5(\sin(\pi(2)))(\sin(\frac{\pi(2)}{4}))(\sin(\frac{\pi(2)^2}{8}))$

$= 50(\sin(\frac{\pi}{2})) + 2(\cos(\frac{\pi}{2}))(\sin(\frac{\pi}{2})) + 5(\cos(2\pi))(\sin(\frac{\pi}{2})) - 5(\sin(2\pi))(\sin(\frac{\pi}{2}))(\sin(\frac{\pi}{2}))$

$= 50 + 5 + 0$

$\boxed{x_p(2) = 55}$

$y_p(2) = 2\sin(\frac{\pi(2)}{4}) + \cos(\frac{\pi(2)}{4}) 5\sin(2\pi)$

$= 2 + 0$

$\boxed{y_p(2) = 2}$

$z_p(2) = 50(\cos(\frac{\pi(2)^2}{8})) - 2(\cos(\frac{\pi(2)}{4}))(\cos(\frac{\pi(2)^2}{8})) + 5(\cos(\pi(2)))(\cos(\frac{\pi(2)^2}{8})) + 5(\sin(\pi(2)))(\sin(\frac{\pi(2)}{4}))(\cos(\frac{\pi(2)^2}{8}))$

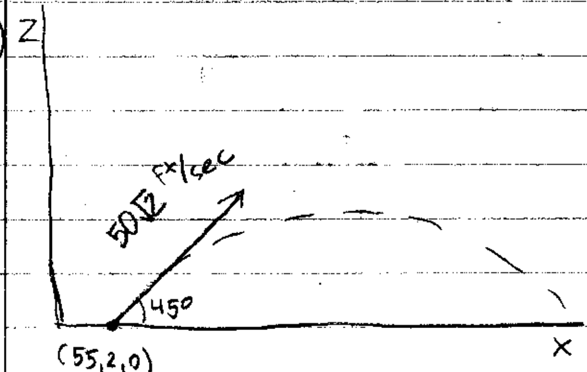
$= 50(\cos(\frac{\pi}{2})) - 2(\cos(\frac{\pi}{2}))(\cos(\frac{\pi}{2})) + 5(\cos(2\pi))(\cos(\frac{\pi}{2})) + 5(\sin(2\pi))(\sin(\frac{\pi}{2}))(\cos(\frac{\pi}{2}))$

$= 0 + 0 + 0$

$\boxed{z_p(2) = 0}$

poop @ time $t = 2$ (or $t = 26$) $\quad (55, 2, 0)$

g)



$V_{0x} = (\cos(45))\, 50\sqrt{2} = 50$ ft/sec

$V_{0z} = (\sin(45))\, 50\sqrt{2} = 50$ ft/sec

$$\vec{a} = \begin{pmatrix} 0 \\ -32 \end{pmatrix} - K \begin{pmatrix} V_x \\ V_z \end{pmatrix} \begin{matrix} = \frac{dv_x}{dt} \\ = \frac{dv_z}{dt} \end{matrix}$$

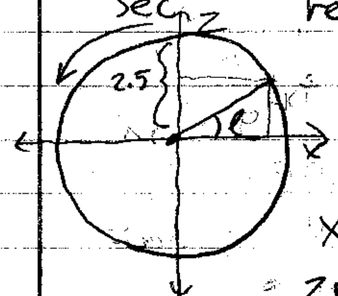
$\frac{dv_x}{dt} = -K v_x \qquad \frac{dv_z}{dt} = -32 - K V_z$

$\frac{dv_x}{v_x} = -K\,dt \qquad \frac{dv_z}{dt} = -K\left(\frac{32}{K} + V_z\right)$

$$\ln|v_x| = -Kt + C_7$$
$$v_x = C_7 e^{-Kt}$$
$$v_x(0) = v_{0x} = C_7$$
$$\frac{dx}{dt} = v_x = 50e^{-Kt}$$
$$x(t) = \left(\frac{50}{-K}\right)e^{-Kt} + C_9$$
$$x(0) = 55 = C_9$$
$$x_c(t) = -500e^{-0.1t} + 55$$

$$\frac{dv_z}{\left(\frac{32}{K} + v_z\right)} = -K\,dt$$
$$\ln\left|\frac{32}{K} + v_z\right| = -Kt + C_8$$
$$\frac{32}{K} + v_z = C_8 e^{-Kt}$$
$$v_z = C_8 e^{-Kt} - \frac{32}{K}$$
$$v_z(0) = v_{0z} = C_8 - \frac{32}{K}$$
$$C_8 = v_{0z} + \frac{32}{K}$$
$$\frac{dz}{dt} = v_z = \left(50 + \frac{32}{K}\right)e^{-Kt} - \frac{32}{K}$$
$$z(t) = \frac{\left(50 + \frac{32}{K}\right)}{-K}e^{-Kt} - \frac{32t}{K} + C_{10}$$
$$z(0) = 0 = C_{10} -$$
$$\boxed{z_c(t) = -500e^{-0.1t} - 3200e^{-0.1t} - 320t}$$

$$\frac{2\ \text{rev}}{\text{sec}}\ \frac{2\pi\ \text{rad}}{\text{rev}} = 4\pi\frac{\text{rad}}{\text{sec}} = \frac{d\ell}{dt}$$
$$\ell = 4\pi t + C_{11}$$
$$\ell(0) = 0 = C_{11}$$



$$x_s = \cos(\ell) \qquad x_s(t) = \cos(4\pi t)$$
$$z_s = \sin(\ell) \qquad z_s(t) = \sin(4\pi t)$$
$$x_p = x_c + x_s$$
$$z_p = z_c + z_s$$

$$\boxed{\begin{aligned} x_p(t) &= \left[-500e^{-0.1t} + 55\right] + \left[\cos(4\pi t)\right] \\ y_p(t) &= 2 \\ z_p(t) &= \left[-500e^{-0.1t} - 3200e^{-0.1t} - 320t\right] + \left[\sin(4\pi t)\right. \end{aligned}}$$

h) $v_z = \left(50 + \frac{32}{K}\right)e^{-Kt} - \frac{32}{K} = 0$

$$50 + 320e^{-Kt} - 320 = 0$$
$$370e^{-Kt} = 320$$
$$e^{-Kt} = \frac{320}{370} \qquad t = \ln\frac{370}{320}$$
$$z(t) = -500e^{-Kt} - 3200e^{-Kt} - 320t$$
$$z_{max} = -500\left(\frac{320}{370}\right) - 3200\left(\frac{320}{370}\right) - 320\left[\ln\left(\frac{370}{320}\right)\right]$$
$$\boxed{= -3200 - 320\left[\ln\left(\frac{370}{320}\right)\right]}$$