Geometric Series

1. Infinite summation of positive quantities

Everyone knows from a very early age that $1/3$ can be written in decimal form as $0.333333\ldots$. Usually, elementary school kids write $0.\overline{3}$ to save space. Here, however, we will expand and analyse "one-third" in more detail. Notice that:

$$0.333333\cdots = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

So really this is an **INFINITE SUMMATION** (or **SERIES**). If at some point in this infinite process we get tired and decide to stop adding any more terms, we get, say $0.3 + 0.03 + 0.003 + 0.0003 = 0.3333$, what is known as a **PARTIAL SUM**. In this example, because the terms that we are adding are all positive, the partial sums increase. It is costumary to write s_n to indicate the summation of the first n terms, or n-th partial sum. So for instance here, 0.3333 is s_4.

In this example the partial sum s_n is the decimal $0.3\ldots3$ with exactly n copies of the number 3 after the decimal point. So the difference between $1/3$ and s_n is equal to $0.0\ldots0\overline{3}$, and it tends to 0 as n tends to infinity.

Definition 1.1. We say that an infinite summation **CONVERGES** if the partial sums s_n tend to a finite limit as n tends to infinity.

The series $0.3 + 0.03 + 0.003 + 0.0003 + \cdots$ is a very special type of series known as **GEOMETRIC SERIES**. In fact, let us rewrite it first using the scientific notation:

$$3 \cdot 10^{-1} + 3 \cdot 10^{-2} + 3 \cdot 10^{-3} + 3 \cdot 10^{-4} + 3 \cdot 10^{-5} + \cdots$$

Using the laws of exponents and factoring 3 out we get

$$3\left[10^{-1} + (10^{-1})^2 + (10^{-1})^3 + (10^{-1})^4 + (10^{-1})^5 + \cdots\right]$$

Notice that we are summing higher and higher powers of a same fixed number: 10^{-1}. This process turns out to be so common and so widely used that it behooves us to make the following definition.

Definition 1.2. Fix a number q of your choice. The process of summing up all the integer powers of q starting from $q^0 = 1$ then adding $q^1 = q$, then q^2, then q^3, etc...

$$1 + q + q^2 + q^3 + q^4 + \cdots + q^n + \cdots$$

is called the **GEOMETRIC SERIES WITH FACTOR** q.

Remark 1.3. In this context the letter n typically represents an arbitrary integer, like $0,\ 1,\ 2$, and so on.

The infinite sum $3\left[10^{-1} + (10^{-1})^2 + (10^{-1})^3 + (10^{-1})^4 + (10^{-1})^5 + \cdots\right]$ does not quite fit with our definition of geometric series. Indeed, we are adding up consecutive powers of $q = 10^{-1}$, but skipped over the first term $q^0 = 1$. There are two ways to manipulate this expression. One would be to add and subtract 1, the other way is to factor out a 10^{-1} term and get that

$$
\begin{aligned}
3\left[10^{-1} + (10^{-1})^2 + (10^{-1})^3 + (10^{-1})^4 + (10^{-1})^5 + \cdots\right] &= \\
3\cdot 10^{-1}\left[10^{-1} + (10^{-1})^2 + (10^{-1})^3 + (10^{-1})^4 + (10^{-1})^5 + \cdots\right]
\end{aligned}
$$

Claim 1.4. *The geometric series with factor q converges when $0 < q < 1$ and does not converge when $q \geq 1$.*

In order to verify this claim we must fix $q > 0$ and try to find out if the partial sums of the geometric series with factor q tend to a finite limit. What are these partial sums? Here are the first few:

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + q \\
s_3 &= 1 + q + q^2 \\
s_4 &= 1 + q + q^2 + q^3 \\
s_5 &= 1 + q + q^2 + q^3 + q^4 \\
\cdots & \quad \cdots \\
s_n &= 1 + q + q^2 + q^3 + q^4 + \cdots + q^{n-1} \\
s_{n+1} &= 1 + q + q^2 + q^3 + q^4 + \cdots + q^{n-1} + q^n
\end{aligned}
$$

The last two lines make sense whenever $n = 1, 2, 3, \ldots$.

The numbers $s_1, s_2, s_3, \ldots$ form a **SEQUENCE**. Sequences can be given in two form: implicitly or explicitly. The implicit form comes as a **RECURRENCE RELATION**, namely we are given the first term of the sequence, say s_1, and also we are given a recipe for passing from one term to the next: $s_{n+1} = F(s_n)$, where F is some function. You should compare this with a differential equation with some given initial condition. The explicit form is $s_n = f(n)$ for some function f, which can be thought as the solution to the differential equation.

The partial sums of the geometric series with factor q can be expressed in recurrence form in a couple of ways: either as $s_{n+1} = s_n + q^n$ or $s_{n+1} = 1 + qs_n$ (with initial condition $s_1 = 1$). The latter recurrence relation should remind you of the differential equation $dy/dx = Ay + B$ that we have studied extensively this semester.

In order to prove Claim 1.4 however we need to find an explicit formula for the partial sums. We start from

$$qs_n = s_{n+1} - 1 = 1 + q + q^2 + q^3 + q^4 + \cdots + q^{n-1} + q^n - 1 = s_n + q_n - 1$$

and solve for s_n to get

$$\boxed{s_n = \frac{1 - q^n}{1 - q}}$$

From this formula we deduce that the partial sums s_n tend to a finite limit only when $0 < q < 1$, and this limit is $1/(1-q))$. We also see that for $q > 1$ the partial sums diverge to $+\infty$, and finally when $q = 1$, all the powers of q are equal to 1, hence $s_n = n$, and this diverges to $+\infty$ as well. Claim 1.4 is proved.

Now we give two applications of the geometric series.

Example 1.5 (**DRUG INTAKE**)**.** Suppose that one dose of a medicine yields an initial concentration C_0 of a certain chemical into the blood-stream, and suppose that this concentration then decays in time with some decay factor $k > 0$. Then

$$C(t) = C_0 e^{-kt}$$

for $t > 0$. However as it is often the case, suppose that the same dose of the medicine is taken repeatedly at intervals of T hours. Then following happens:

$$\begin{array}{ll} t = 0 & C(0) = C_0 \\ t = T & C(T) = C_0 e^{-kT} + C_0 \\ t = 2T & C(2T) = C_0 e^{-k2T} + C_0 e^{-kT} + C_0 \\ t = 3T & C(3T) = C_0 e^{-k3T} + C_0 e^{-k2T} + C_0 e^{-kT} + C_0 \\ \cdots & \cdots \\ t = nT & C(nT) = C_0 e^{-knT} + C_0 e^{-k(n-1)T} + \cdots C_0 \end{array}$$

We recognize $C(nT)$ has the $n+1$ partial sum of the geometric series with factor $q = e^{-kT}$. So because of the boxed formula above:

$$C(nT) = C_0 \frac{1 - e^{-k(n+1)T}}{1 - e^{-kT}}$$

As n tends to infinity the concentration $C(nT)$ tends to the *Stable Concentration*:

$$C_\infty = \frac{C_0}{1 - e^{-kT}}$$

Example 1.6 (COIN FLIPPING). A good model for random phenomena is provided by the science of coin flipping. Often the coin is taken to be biased so that Heads (H) has probability p for some $0 < p < 1$ (e.g $p = 1/3$). Tails (T) must then have probability $q = 1 - p$. Suppose now that we want to know the expected number of throws required in order to get Heads. The usual way to proceed is to list all the possible outcomes and introduce a random variable X which to each outcome assign a number namely the number of throws:

$$\begin{array}{ll} H & X = 1 \\ TH & X = 2 \\ TTH & X = 3 \\ TTTH & X = 4 \\ TTTTH & X = 5 \\ \dots & \dots \end{array}$$

This yields a *discrete* probability distribution on the set of numbers $\{1, 2, 3, \dots\}$. In fact:

$$\begin{array}{ll} P(X = 1) & = P(H) = p \\ P(X = 2) & = P(T)P(H) = qp \\ P(X = 3) & = P(T)P(T)P(H) = q^2 p \\ P(X = 4) & = P(T)P(T)P(T)P(H) = q^3 p \\ \dots & \dots \\ P(X = n) & = q^{n-1} p \end{array}$$

($P(TH) = P(T)P(H)$ because the two events are independent).

We can verify that this is indeed a probability distribution by adding all the weights up and using Claim 1.4:

$$p + qp + q^2 p + q^3 p + \cdots = p[1 + q + q^2 + q^3 + \cdots] = p\frac{1}{1-q} = p\frac{1}{p} = 1$$

In order to find the expected number of throws we need to compute $E(X)$ which is a weighted average of the possible lengths with the weights given above:

$$\begin{aligned} E(X) &= 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) + 4 \cdot P(X = 4) + \cdots \\ &= p + 2qp + 3q^2 p + 4q^3 p + \cdots + nq^{n-1} p + \cdots \end{aligned}$$

After inspection we realize that this series is NOT a geometric series because of the coefficient n in front of q^{n-1}. Therefore we cannot use Claim 1.4 rightaway to compute this sum. We first will need to develop more tools.

The first remark is that the proof of Claim 1.4 works just as well when q is negative. So we can reformulate the claim as follows.

Theorem 1.7. *When $-1 < q < 1$ the infinite sum*

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots$$

equals

$$\frac{1}{1-q}.$$

This is no longer true when $q \leq -1$ or $q \geq 1$.

Notice that there is a whole interval of values of q for which the geometric series with factor q converges and this is the symmetric interval about the origin $(-1, 1)$.

We now turn things around and start from the function

$$f(x) = \frac{1}{1-x}$$

instead. The graph of $y = f(x)$ is the usual hyperbola in the first and third quadrant, shifted by one unit to the right. So in $x = 1$ there is a vertical asymptote. The fact that for $-1 < x < 1$ the identity

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \tag{1.1}$$

holds is interpreted as an **APPROXIMATION SCHEME**. Namely the partial sums $s_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$ are polynomials of degree n which for x fixed in $(-1, 1)$ converge to $f(x)$ as n tends to infinity. The larger n is taken the better the approximation.

It is a fact that will not be proved in this class that one can differentiate the infinite sum on the right-hand side of equation (1.1) **TERM-BY-TERM** and get a new infinite sum which converges to $f'(x)$. More specifically, we have for $-1 < x < 1$

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \tag{1.2}$$

Proving that one can differentiate this infinite sum term-by-term entails showing that it is OK to switch the order in which two limits are performed. In fact, taking deriatives is done by taking the limit of the difference quotient, and infinite sums are the limit of the partial sums: these are the two limits involved.

Example 1.8. We can now return to Example 1.6 and compute the expected number of throws for Heads to come out. Recall that this is

$$E(X) = p + 2pq + 3pq^2 + 4pq^3 + \cdots + npq^{n-1} + \cdots$$

Therefore factoring p out and using the identity (1.2) for $x = q$ we get

$$E(X) = p(1 + 2q + 3q^2 + \cdots + nq^{n-1} + \cdots) = p\frac{1}{(1-q)^2} = \frac{1}{p}$$

since $p = 1 - q$. Recall that p is the probability that when flipping the biased coin the result is Heads. Suppose then that p is very small, since $E(X) = 1/p$ the expected number of throws before Heads turns up is large.

The next thing one would like to compute is the variance of X: $Var(X) = E(X^2) - E(X)^2$ (the second moment minus the first moment square). Since $E(X) = 1/p$ we only need to compute the second moment

$$E(X^2) = 1^2 \cdot P(X = 1) + 2^2 \cdot P(X = 2) + 3^2 \cdot P(X = 3) + \cdots + n^2 \cdot P(X = n) + \cdots$$

Once again we can use the values of $P(X = n)$ and get

$$\begin{aligned} E(X^2) &= p + 2^2pq + 3^2pq^2 + \cdots n^2pq^{n-1} + \cdots \\ &= p(1 + 2^2q + 3^2q^2 + 4^2q^3 + \cdots + n^2q^{n-1} + \cdots \end{aligned}$$

This last series in terms of q sums increasing powers of q which are multiplied by coefficients that grow like n^2. This is not like anything we have seen before, yet it is also not very far. In fact, consider differentiating (1.2), i.e., differentiate the geometric series in (1.1) twice, we get

$$f''(x) = \frac{2}{(1-x)^3} = 2 + (3 \cdot 2)x + (4 \cdot 3)x^2 + \cdots + (n \cdot (n-1))x^{n-2} + \cdots$$

Here is where the **SIGMA NOTATION** comes in handy. The generic term in the last equation is $(n \cdot (n-1))x^{n-2}$ where $n = 2, 3, 4, \ldots$. So we can write the identity as

$$\sum_{n=2}^{+\infty} (n \cdot (n-1))x^{n-2} = \frac{2}{(1-x)^3}$$

Expanding the product and factoring $1/x$ out we get

$$x^{-1}\left(\sum_{n=2}^{+\infty} n^2x^{n-1} - nx^{n-1}\right) = \frac{2}{(1-x)^3}$$

If you are worried about what just happened (factoring x^{-1} out) notice that here x is fixed and therefore can be treated as a constant, it's n that varies and tends to infinity!

It is another fact that will not be proved in this class that the sum above can be split into two sums. Namely, after multiplying both sides by x and splitting the infinite sum we can write

$$\sum_{n=2}^{+\infty} n^2 x^{n-1} - \sum_{n=2}^{+\infty} n x^{n-1} = \frac{2x}{(1-x)^3} \tag{1.3}$$

First look at the second sum and write out the first few terms:

$$\sum_{n=2}^{+\infty} n x^{n-1} = 2x + 3x^2 + 4x^3 + \cdots$$

This can easily be recognized as the sum in (1.2), except that the first term is missing, so we just add and subtract 1 and get

$$\sum_{n=2}^{+\infty} n x^{n-1} = 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2} - 1$$

Onthe other hand, writing out the first few terms of the first sum in (1.3) we get

$$2^2 x + 3^2 x^2 + 4^2 x^3 + 5^2 x^4 + \cdots$$

Again we are missing the intial 1 which we can add and subtract. So finally

$$\begin{aligned} 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + 5^2 x^4 + \cdots &= 1 + \sum_{n=2}^{+\infty} n^2 x^{n-1} \\ &= 1 + \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2} - 1 \\ &= \frac{2x + (1-x)}{(1-x)^3} = \frac{x+1}{(1-x)^3}. \end{aligned}$$

We can now use this formula to compute

$$\begin{aligned} E(X^2) &= p(1 + 2^2 q + 3^2 q^2 + \cdots + n^2 q^{n-1} + \cdots \\ &= p\frac{(1-p)+1}{p^3} = \frac{2-p}{p^2} \end{aligned}$$

and finally

$$Var(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

2. Expanding functions into infinite sums

We now return to the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \tag{2.1}$$

which holds for $-1 < x < 1$, and which can be interpreted as an expansion of the function $f(x) = 1/(1-x)$ in the interval $(-1, 1)$. We have seen above that this identity can be differentiated to obtain for $-1 < x < 1$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \tag{2.2}$$

and if we differentiate once more we get for $-1 < x < 1$

$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots + n(n-1)x^{n-2} + \cdots \tag{2.3}$$

More generally, if we differentiate $k = 1, 2, 3, 4, \ldots$ times:

$$\frac{k!}{(1-x)^{k+1}} = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} x^{n-k} \tag{2.4}$$

whenever $-1 < x < 1$. The notation $k!$ means $k \cdot (k-1) \cdot (k-2) \cdots 2$ and is read "k **FACTORIAL**".

Exercise 2.1. Check that putting $k = 1$ and $k = 2$ in (2.4) we recover the identities (2.2) and (2.3).

On the other hand, one can also integrate (2.2) term-by-term and obtain for $-1 < x < 1$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1} \tag{2.5}$$

The constant of integration is found to be zero by letting $x = 0$ in both sides of the equation. This gives me the opportunity to expand on an old theme. Recall the function $y = \ln(1+x)$ whose graph is just the graph of the logarithm shifted one unit to the left (so that the vertical asymptote is at $x = -1$). At the beginning of the semester we used the tangent line approximation to estimate $\ln(1+x) \approx x$ for x small. So for instance we could say that $\ln(1.1) = \ln(1+0.1) \approx 0.1$. Now we can say a lot more and get approximations of arbitrarily high order. In fact, plugging in $-x$ in (2.5) and multiplying both sides by -1 we get that for $-1 < x < 1$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} \tag{2.6}$$

Thus for instance this is the third order approximation of ln(1.1):

$$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} = 0.1 - 0.005 + 0.000\overline{3}$$

The alert reader has by now noticed that in all these expansions the interval of x for which the series converge is always symetric about the origin and the width of the interval is determined by the vertical asymptotes of the function that is being expanded. It is therefore natural to ask: *Is it true that the interval of convergence is always determinated by the vertical asymptote?* The answer to this question is "yes and no". The next example shows that vertical asymptotes have little to do with the interval of convergence, yet...

Consider

$$f(x) = \frac{1}{1+x^2}$$

Notice that there are no vertical asymptotes for this function, yet the interval of convergence is still $(-1, 1)$. In fact, plug in $-x^2$ into equation (2.1) and get that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \tag{2.7}$$

whenever $-1 < -x^2 < 1$, which happens if and only if $-1 < x < 1$. The mistery of this interval of convergence in the absence of vertical asymptotes can be explained using **COMPLEX NUMBER**. Probably the most famous complex number is i, "the square-root of minus one". As the name indicates i has the property that $i^2 = -1$. Also if the real numbers are all the point in the plane with coordinates $(x, 0)$, then i can be thought as being the point $(0, 1)$ on the y-axis. Notice then that

$$(x-i)(x+i) = x^2 - i^2 = x^2 + 1$$

So using partial fractions;

$$\frac{1}{1+x^2} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

This shows that indeed, if we alllow x to also wander around in the plane, there is a vertical asymptote at i and $-i$ which explains why the interval of convergence is just $(-1, 1)$.

We finish with the expansion of another function that we made our acquaintance with recently: the "arctangent". Just integrate equation (2.7) and get

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \tag{2.8}$$