CALCULUS II – FINAL – Fall 2003
NO CALCULATORS – SHOW WORK

NAME KEY
Rec. Instr. __________
Rec. Time. __________

1.(25pts) Integrate:

a. $\displaystyle\int_2^{+\infty} \frac{1}{x^2+x-2}dx =$

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{Ax+2A+Bx-B}{(x-1)(x+2)}$$

$$\Delta = 1-(-8) = 9 > 0$$

$$2 \text{ ROOTS } x = \frac{-1\pm 3}{2} \begin{cases} -2 \\ 1 \end{cases}$$

$$\frac{(A+B)x + (2A-B)}{x^2+x-2} = \frac{1}{x^2+x-2}$$

$$\begin{cases} A+B=0 \Rightarrow A=-B & \Rightarrow B=-1/3 \\ 2A-B=1 \Rightarrow 2A+A=3A=1 \Rightarrow A=\frac{1}{3} \end{cases}$$

$$\frac{1}{3}\int_2^{+\infty}\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx = \frac{1}{3}\left[\ln|x-1|-\ln|x+2|\right]\Big|_2^{+\infty} = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\Big|_2^{+\infty}$$

$$= \frac{1}{3}\left[\lim_{x\to+\infty} \cancel{\ln\left|\frac{x-1}{x+2}\right|}^{0}\right] - \frac{1}{3}\ln\left|\frac{1}{4}\right| = \boxed{\frac{1}{3}\ln 4}$$

b. $\displaystyle\int_0^{\infty} te^{-2t}dt$

D: $t \to 1$, I: $e^{-2t} \to -\frac{1}{2}e^{-2t}$

$$= -\frac{t}{2}e^{-2t}\Big|_0^{+\infty} - \int_0^{+\infty} -\frac{1}{2}e^{-2t}dt = \lim_{t\to+\infty}\left(-\frac{t}{2}e^{-2t}\right) - \frac{1}{4}e^{-2t}\Big|_0^{+\infty} = \boxed{\frac{1}{4}}$$

c. $y' = 2-3y$ **and** $y(0) = 1$

$$y' = -3\left(y-\frac{2}{3}\right)$$

$$\int \frac{y'(x)dx}{y-\frac{2}{3}} = \int -3\,dx \Rightarrow \ln\left|y-\frac{2}{3}\right| = -3x+c \Rightarrow y-\frac{2}{3} = \pm e^c \cdot e^{-3x}$$

$$@\ x=0 \quad 1-\frac{2}{3} = \frac{1}{3} = \pm e^c \Rightarrow \boxed{y = \frac{2}{3}+\frac{1}{3}e^{-3x}}$$

2.(25pts) (a) Expand the function $f(x) = e^{2x}\arctan(3x)$ at 0 to the third degree (i.e. up to terms involving x^3).

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \Rightarrow e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$(\arctan x)' = \frac{1}{1+x^2} = 1 - x^2 + \cdots \Rightarrow \arctan x = x - \frac{x^3}{3} + \cdots$$

$$\Rightarrow \arctan(3x) = 3x - 9x^3 + \cdots$$

$$f(x) = 3x + 6x^2 + 6x^3 - 9x^3 + \cdots = \boxed{3x + 6x^2 - 3x^3 + \cdots}$$

(b) Expand the function $f(x) = \ln\dfrac{2}{x+1}$ in a power series centered at $x = 2$ (Hint: consider $f'(x)$). What is the radius of convergence?

$$f'(x) = -\frac{1}{(x+1)} \quad \left(\text{because } f(x) = \ln 2 - \ln(x+1)\right).$$

$$= -\frac{1}{(x-2)+3} = -\frac{1}{3}\frac{1}{1-\left[-\frac{(x-2)}{3}\right]} = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(x-2)^n$$

INTEGRATE TERM-BY-TERM: $f(x) = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\frac{(x-2)^{n+1}}{n+1} + C$

PLUG $x=2$ $\quad f(2) = \ln\frac{2}{3} = C \quad \left((2-2)^{n+1} = 0 \text{ for } n=0,1,2,\ldots\right)$

So $\boxed{f(x) = \ln\frac{2}{3} - \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}\frac{(x-2)^{n+1}}{n+1}}$

$$= \ln\frac{2}{3} - \frac{1}{3}(x-2) + \frac{1}{18}(x-2)^2 - \frac{1}{81}(x-2)^3 + \cdots$$

3. (25pts) (Movie logistics) Movie attendance spreads like a disease. Assume that 1 million people are expected to be potentially interested in the movie "The Return". If after 2 days 100,000 people have seen the movie and after 10 days a total of 200,000 people have seen the movie, after how many day will 500,000 people have seen the movie? (Use the equation $dy/dt = ky(M - y)$ and use $\ln 2 = 0.7,\ \ln 9 = 2.2$).

$M = 1$ million $\quad \frac{dy}{dt} = ky(M-y)$

$$\int \frac{dy}{y(M-y)} = \int k\,dt \Rightarrow \frac{1}{M}\int \left(\frac{1}{M-y} + \frac{1}{y}\right) dy = \int k\,dt.$$

$$-\ln|M-y| + \ln|y| = Mkt + c = \ln\left|\frac{y}{M-y}\right|$$

$$\frac{y}{M-y} = \pm e^c \cdot e^{Mkt}.$$

① $t=2 \quad y(2) = 0.1M \Rightarrow \frac{0.1M}{M-0.1M} = \frac{0.1}{0.9} = \frac{1}{9} = \pm e^c e^{Mk2}.$

$$\Rightarrow \pm e^c = \frac{1}{9}e^{-Mk2} \Rightarrow \frac{y}{M-y} = \frac{1}{9}e^{-Mk2}e^{Mkt}$$

② $t=10 \quad y(10) = 0.2M \Rightarrow \frac{0.2M}{M-0.2M} = \frac{1}{9}e^{-Mk2}e^{Mk10} = \frac{1}{9}e^{Mk8}$

$$\Rightarrow \frac{0.2}{0.8} = \frac{1}{4} = \frac{1}{9}e^{Mk8} \Rightarrow Mk8 = \ln\frac{9}{4} \Rightarrow Mk = \frac{1}{8}\ln\frac{9}{4}$$

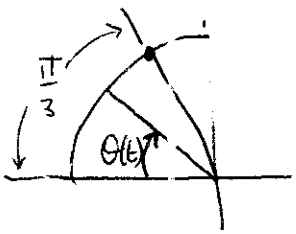
when $y = 0.5M \Rightarrow \frac{0.5M}{M-0.5M} = 1 = \frac{1}{9}e^{-Mk2}e^{Mkt} \Rightarrow e^{Mkt} = 9e^{Mk2}$

So $Mkt = \ln 9 + Mk2 \Rightarrow t - 2 = \frac{\ln 9}{Mk} \Rightarrow t = 2 + 8\frac{\ln 9}{\ln 9 - 2\ln 2} =$

$$= 2 + 8\frac{2.2}{2.2 - 1.4} = 2 + 8\frac{2.2}{0.8} = \boxed{24}$$

4.(25pts) Basketball legend Kareem Abdul Jabbar's sky hook. His arm is $3ft$ long and rotates about his shoulder which is centered at $(0,6)$ in the xy-plane. Assume the movement starts ($t=0$) when the arm is horizontal (so that the basketball is initially at $(-3,6)$, at rest) and then swings clockwise by a $60°$ angle. Letting θ be the angle measured clockwise with $\theta(0)=0$, assume that the angular acceleration $d^2\theta/dt^2$ is constant and is equal to $\frac{3\pi}{2} rad/sec^2$.

(a) Find the time T when $\theta(T)=\pi/3$.

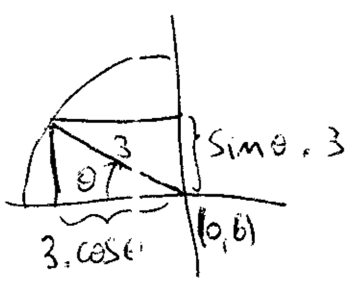


$\theta(0)=0$ & $\left.\frac{d\theta}{dt}\right|_{t=0}=0.$

$\frac{d^2\theta}{dt^2}=\frac{3\pi}{2} \Rightarrow \frac{d\theta}{dt}=\frac{3\pi}{2}t+\cancel{c}^{0} \Rightarrow \theta=\frac{3\pi}{4}t^2+\cancel{c}^{0}$

$\theta(T)=\frac{\pi}{3}=\frac{3\pi}{4}T^2 \Rightarrow T^2=\frac{4}{9} \Rightarrow \boxed{T=\frac{2}{3}}$

(b) Write the equations for the position $(x(t),y(t))$ of the ball at time t as it swings in Kareem's hand.



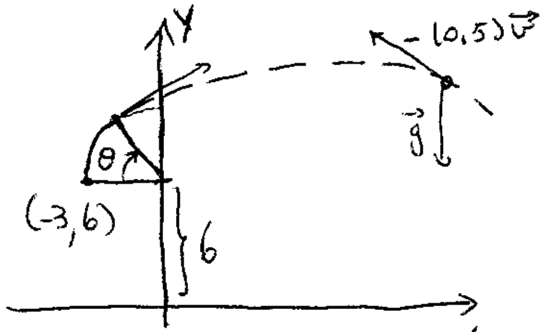
$\begin{cases} X=0-3\cos\theta \\ Y=6+3\sin\theta \end{cases} \Rightarrow \begin{cases} X(t)=-3\cos\left(\frac{3\pi}{4}t^2\right) \\ Y(t)=6+3\sin\left(\frac{3\pi}{4}t^2\right) \end{cases}$

(c) Determine the position $(x(T),y(T))$ and velocity $(v_x(T),v_y(T))$ at time $t=T$ (T is from part (a)).

$\begin{cases} v_x(t)=\frac{dx}{dt}=\frac{9\pi}{4}.2t\sin\left(\frac{3\pi}{4}t^2\right) \\ v_y(t)=\frac{dY}{dt}=\frac{9\pi}{4}.2t\cos\left(\frac{3\pi}{4}t^2\right) \end{cases}$ Since $\frac{3\pi}{4}T^2=\frac{\pi}{3}$ we get

$\begin{cases} v_x(T)=\frac{9\pi}{4}.2.\frac{2}{3}\sin\left(\frac{\pi}{3}\right)=\frac{3\pi\sqrt{3}}{2} \\ v_y(T)=\frac{9\pi}{4}2.\frac{2}{3}\cos\left(\frac{\pi}{3}\right)=\frac{3\pi}{2} \end{cases}$ & $\begin{cases} X(T)=-3\cos\left(\frac{\pi}{3}\right)=-\frac{3}{2} \\ Y(T)=6+3\sin\left(\frac{\pi}{3}\right)=6+\frac{3\sqrt{3}}{2} \end{cases}$

(d) **Time $t = T$ is when the ball leaves Kareem's hand. It then flies subject only to gravity $(0, -32)$ and to air resistance $-(0.5)\vec{v}$. Write equations for the motion of the ball for time $t \geq T$ using the initial conditions in part (c). (Assume the mass of the ball to be 1.)**



$$\frac{d\vec{v}}{dt} = \vec{g} - (0.5)\vec{v} \quad (\text{Eq. of motion})$$

$$\begin{cases} \dfrac{dv_x}{dt} = -0.5\, v_x \\ \dfrac{dv_y}{dt} = -32 - 0.5\, v_y = -0.5(v_y + 64) \end{cases} \Rightarrow \begin{cases} \dfrac{dv_x}{v_x} = -0.5\,dt \\ \dfrac{dv_y}{v_y+64} = -0.5\,dt. \end{cases}$$

$$\begin{cases} \ln|v_x| = -0.5t + c_1 \\ \ln|v_y + 64| = -0.5t + c_2 \end{cases} \Rightarrow \begin{cases} v_x = \pm e^{c_1} \cdot e^{-0.5t} \\ v_y + 64 = \pm e^{c_2} \cdot e^{-0.5t} \end{cases}$$

@ $t = T = 2/3$

$$\begin{cases} 3\pi\frac{\sqrt{3}}{2} = \pm e^{c_1} e^{-0.5\cdot\frac{2}{3}} \\ \frac{3\pi}{2} + 64 = \pm e^{c_2} e^{-0.5\cdot\frac{2}{3}} \end{cases} \Rightarrow \begin{cases} \pm e^{c_1} = 3\pi\frac{\sqrt{3}}{2} e^{\frac{1}{3}} = A \\ \pm e^{c_2} = \left(\frac{3\pi}{2} + 64\right) e^{\frac{1}{3}} = B \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = A e^{-0.5t} \\ \dfrac{dy}{dt} = B e^{-0.5t} - 64. \end{cases} \Rightarrow \begin{cases} x = \dfrac{A}{-0.5} e^{-0.5t} + C_1 \\ y = \dfrac{B}{-0.5} e^{-0.5t} - 64t + C_2 \end{cases}$$

@ $t = T = \frac{2}{3}$

$$\begin{cases} -\frac{3}{2} = -2 \cdot 3\pi \frac{\sqrt{3}}{2} e^{1/3} \cdot e^{-1/3} + C_1 \quad \Rightarrow C_1 = 3\sqrt{3}\pi - \frac{3}{2} \\ 6 + \frac{3\sqrt{3}}{2} = -2\left(\frac{3\pi}{2} + 64\right) e^{1/3} e^{-1/3} - 64 \cdot \frac{2}{3} + C_2 \end{cases}$$

$$\Rightarrow C_2 = 3\pi + \frac{3\sqrt{3}}{2} + 6 + 128 + \frac{128}{3}$$

$$\begin{cases} x(t) = -3\sqrt{3}\pi e^{1/3} e^{-0.5t} + 3\sqrt{3}\pi - \frac{3}{2} \\ y(t) = -(3\pi + 128) e^{1/3} e^{-0.5t} - 64t + 3\pi + \frac{3\sqrt{3}}{2} + \frac{530}{3} \end{cases}$$

5. **BONUS PROBLEM (5pts Extra Points) Consider the function**

$$F(x) = \int_0^x e^{-t^2}\,dt$$

Find the antiderivative of F. (The solution may involve the function F itself.)

INTEGRATION BY PARTS.

$$\int 1 \cdot F(x)\,dx = xF(x) - \int x e^{-x^2}\,dx$$

I: $1 \to x$ D: $F(x) \to e^{-x^2}$

$$= xF(x) + \frac{1}{2}\int (-2x) e^{-x^2}\,dx$$

$$= xF(x) + \frac{1}{2} e^{-x^2} + C$$